

# Sharp large deviations for the fractional Ornstein-Uhlenbeck process

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## Abstract

We investigate the sharp large deviation properties of the energy and the maximum likelihood estimator for the Ornstein-Uhlenbeck process driven by a fractional Brownian motion with Hurst index greater than one half.

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## 1 Introduction.

Since the pioneer works of Kolmogorov, Hurst and Mandelbrot, a wide range of literature is available on the statistical properties of Fractional Brownian Motion (FBM). On the other hand, one can realize that its large deviations properties were not deeply investigated. Our purpose is to establish sharp large deviations for functionals associated with the Ornstein-Uhlenbeck process driven by a fractional Brownian motion

$$(1.1) \quad dX_t = \theta X_t dt + dW_t^H$$

where the initial state  $X_0 = 0$  and the drift parameter  $\theta$  is strictly negative. The process  $(W_t^H)$  is a FBM with Hurst parameter  $0 < H < 1$  which means that  $(W_t^H)$  is a Gaussian process with continuous paths such that  $W_0^H = 0$ ,  $\mathbb{E}[W_t^H] = 0$  and

$$\mathbb{E}[W_t^H W_s^H] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).$$

The weighting function  $w$  defined, for all  $0 < s < t$ , by  $w(t, s) = w_H^{-1} s^{-H+1/2} (t-s)^{-H+1/2}$  where  $w_H$  is a normalizing positive constant, plays a fundamental role for stochastic

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calculus associated with  $(W_t^H)$ . In particular, for  $t > 0$ , let

$$(1.2) \quad M_t = \int_0^t w(t, s) dW_s^H.$$

It was proven by Norros *et al* [12] page 578 that  $(M_t)$  is a Gaussian martingale with quadratic variation  $\langle M \rangle_t = \lambda_H^{-1} t^{2-2H}$  where

$$\lambda_H = \frac{8H(1-H)\Gamma(1-2H)\Gamma(H+1/2)}{\Gamma(1/2-H)}$$

and  $\Gamma$  stands for the classical gamma function. In addition, it is also more convenient to study the behavior of

$$(1.3) \quad Y_t = \int_0^t w(t, s) dX_s = \theta \int_0^t w(t, s) X_s ds + M_t.$$

It was shown by Kleptsyna and Le Breton [10] that whenever,  $H > 1/2$ , relation (1.3) can be rewritten as

$$(1.4) \quad Y_t = \theta \int_0^t Q_s d\langle M \rangle_s + M_t$$

where the process  $(Q_t)$  satisfies for all  $t > 0$

$$Q_t = \frac{l_H}{2} \left( t^{2H-1} Y_t + \int_0^t s^{2H-1} dY_s \right)$$

with  $l_H = \lambda_H / (2(1-H))$ . Consequently, we shall assume in all the sequel that  $H > 1/2$ . It follows from (1.4) that the score function, which is the derivative of the log-likelihood function from observations over the interval  $[0, T]$ , is given by

$$\Sigma_T(\theta) = \int_0^T Q_t dY_t - \theta \int_0^T Q_t^2 d\langle M \rangle_t.$$

Via a similar approach as the one of [3] for the Ornstein-Uhlenbeck process, we shall investigate the large deviation properties for random variables associated with (1.1) such as the energy

$$(1.5) \quad S_T = \int_0^T Q_t^2 d\langle M \rangle_t$$

as well as the maximum likelihood estimator of  $\theta$ , solution of  $\Sigma_T(\theta) = 0$ , explicitly given by

$$(1.6) \quad \hat{\theta}_T = \frac{\int_0^T Q_t dY_t}{\int_0^T Q_t^2 d\langle M \rangle_t}.$$

We also wish to mention the recent work of Bishwal [4] concerning the large deviation properties of the log-likelihood ratio

$$(\theta_0 - \theta_1) \int_0^T Q_t dY_t - \frac{(\theta_0^2 - \theta_1^2)}{2} \int_0^T Q_t^2 d\langle M \rangle_t.$$

As usual, we shall say that a family of real random variables  $(Z_T)$  satisfies a Large Deviation Principle (LDP) with rate function  $I$ , if  $I$  is a lower semicontinuous function from  $\mathbb{R}$  to  $[0, +\infty]$  such that, for any closed set  $F \subset \mathbb{R}$ ,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P}(Z_T \in F) \leq - \inf_{x \in F} I(x),$$

while for any open set  $G \subset \mathbb{R}$ ,

$$-\inf_{x \in G} I(x) \leq \liminf_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P}(Z_T \in G).$$

Moreover,  $I$  is a good rate function if its level sets are compact subsets of  $\mathbb{R}$ . We refer the reader to the excellent book by Dembo and Zeitouni [7] on the theory of large deviations, see also [2], [9]. A classical tool for proving an LDP for  $S_T$  and  $\widehat{\theta}_T$  is the normalized cumulant generating function

$$(1.7) \quad \mathcal{L}_T(a, b) = \frac{1}{T} \log \mathbb{E}[\exp(\mathcal{Z}_T(a, b))]$$

where, for any  $(a, b) \in \mathbb{R}$ ,

$$(1.8) \quad \mathcal{Z}_T(a, b) = a \int_0^T Q_t dY_t + b \int_0^T Q_t^2 d\langle M \rangle_t.$$

The random variable  $\mathcal{Z}_T(a, b)$  allows us an unified presentation of our results. In fact, in order to establish an LDP for  $S_T$  and  $\widehat{\theta}_T$ , it is enough to prove an LDP for  $\mathcal{Z}_T(0, a)$  and  $\mathcal{Z}_T(a, -ca)$ , respectively. The following lemma provides an asymptotic expansion for  $\mathcal{L}_T$ . It enlightens the role of the limit  $\mathcal{L}$  of  $\mathcal{L}_T$  for the LDP, as well as the first order terms  $\mathcal{H}$  and  $\mathcal{K}_T$  for the sharp LDP. One can observe that it is the keystone of all our results.

**Lemma 1.** *Let  $\Delta_H$  be the effective domain of the limit  $\mathcal{L}$  of  $\mathcal{L}_T$*

$$\Delta_H = \left\{ (a, b) \in \mathbb{R}^2 / \theta^2 - 2b > 0 \text{ and } \sqrt{\theta^2 - 2b} > \max(a + \theta; -\delta_H(a + \theta)) \right\}$$

where  $\delta_H = (1 - \sin(\pi H))/(1 + \sin(\pi H))$ . Then, for any  $(a, b)$  in the interior of  $\Delta_H$ , if  $\varphi(b) = \sqrt{\theta^2 - 2b}$ ,  $\tau(a, b) = \varphi(b) - (a + \theta)$  and  $r_T(b) = r_H(\varphi(b)T/2) \exp(-T\varphi(b)) - 1$ , we have the decomposition

$$(1.9) \quad \mathcal{L}_T(a, b) = \mathcal{L}(a, b) + \frac{1}{T} \mathcal{H}(a, b) + \frac{1}{T} \mathcal{K}_T(a, b) + \frac{1}{T} \mathcal{R}_T(a, b)$$

where

$$(1.10) \quad \mathcal{L}(a, b) = -\frac{1}{2}(a + \theta + \varphi(b)),$$

$$(1.11) \quad \mathcal{H}(a, b) = -\frac{1}{2} \log \left( \frac{\tau(a, b)}{2\varphi(b)} \right),$$

$$(1.12) \quad \mathcal{K}_T(a, b) = -\frac{1}{2} \log \left( 1 + \frac{(2\varphi(b) - \tau(a, b))}{2\varphi(b)} r_T(b) \right).$$

Here,  $I_H$  is the modified Bessel function of the first kind [11] and the function  $r_H$  is defined for all  $z \in \mathbb{C}$  with  $|\arg z| < \pi$  by

$$r_H(z) = \frac{\pi z}{\sin(\pi H)} \left( I_H(z)I_{1-H}(z) + I_{-H}(z)I_{H-1}(z) \right).$$

Finally, the remainder is

$$(1.13) \quad \mathcal{R}_T(a, b) = -\frac{1}{2} \log \left( 1 + \frac{(2\varphi(b) - \tau(a, b))^2}{\tau(a, b)(2\varphi(b) + r_T(b)(2\varphi(b) - \tau(a, b)))} e^{-2T\varphi(b)} \right).$$

**Proof.** The proof is given in Appendix A.  $\square$

**Remark 1.** By use of the duplication formula for the gamma function [11], one can realize that if  $H = 1/2$ ,  $r_H(z) = e^{2z} + e^{-2z}$  which immediately leads to  $r_T(b) = e^{-2T\varphi(b)}$ . Consequently, in that particular case,  $\mathcal{K}_T(a, b)$  as well as  $\mathcal{R}_T(a, b)$  go exponentially fast to zero and we find again Lemma 2.1 of [3] which is the keystone for all results in [3].

## 2 The energy.

First of all, we shall focus our attention on the energy ( $S_T$ ). One can observe that the strong law of large numbers as well as the Central Limit Theorem (CLT) for the sequence  $(S_T)$  were not previously established in the literature.

**Proposition 2.** We have

$$(2.1) \quad \lim_{T \rightarrow \infty} \frac{S_T}{T} = -\frac{1}{2\theta} \quad a.s.$$

Moreover, we also have the CLT

$$(2.2) \quad \frac{1}{\sqrt{T}} \left( S_T + \frac{T}{2\theta} \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, -\frac{1}{2\theta^3} \right).$$

**Proof of Proposition 2.** The almost sure convergence (2.1) clearly follows from Theorem 3 below as the sequence  $(S_T/T)$  satisfies an LDP with good rate function  $I$  given by (2.11). It is not hard to see that  $I(c) = 0$  if and only if  $c = -1/2\theta$ . In order to prove the CLT given by (2.2), denote

$$V_T = \frac{1}{\sqrt{T}} \left( S_T + \frac{T}{2\theta} \right).$$

For all  $a \in \mathbb{R}$ , let  $\Lambda_T(a) = \mathbb{E}[\exp(aV_T)]$ . We clearly have

$$\Lambda_T(a) = \exp \left( \frac{a\sqrt{T}}{2\theta} \right) \mathbb{E} \left[ \exp \left( \frac{aS_T}{\sqrt{T}} \right) \right].$$

Hence, we deduce from the decomposition (1.9) with  $a = 0$  and  $b = a$  that

$$(2.3) \quad \Lambda_T(a) = \exp \left( \frac{a\sqrt{T}}{2\theta} + T\mathcal{L} \left( 0, \frac{a}{\sqrt{T}} \right) + \mathcal{H} \left( 0, \frac{a}{\sqrt{T}} \right) + \mathcal{K}_T \left( 0, \frac{a}{\sqrt{T}} \right) + \mathcal{R}_T \left( 0, \frac{a}{\sqrt{T}} \right) \right).$$

On the one hand,

$$\mathcal{L}\left(0, \frac{a}{\sqrt{T}}\right) = -\frac{1}{2}(\theta + \varphi_T)$$

where

$$\varphi_T = \sqrt{\theta^2 - \frac{2a}{\sqrt{T}}} = -\theta \sqrt{1 - \frac{2a}{\theta^2 \sqrt{T}}}.$$

Consequently, as

$$\varphi_T = -\theta + \frac{a}{\theta \sqrt{T}} + \frac{a^2}{2\theta^3 T} + o\left(\frac{1}{T}\right)$$

we obtain that

$$(2.4) \quad \lim_{T \rightarrow \infty} \frac{a\sqrt{T}}{2\theta} + T\mathcal{L}\left(0, \frac{a}{\sqrt{T}}\right) = -\frac{a^2}{4\theta^3}.$$

On the other hand, as  $(\varphi_T)$  converges to  $-\theta$ , one can check that

$$(2.5) \quad \lim_{T \rightarrow \infty} \left( \mathcal{H}\left(0, \frac{a}{\sqrt{T}}\right) + \mathcal{K}_T\left(0, \frac{a}{\sqrt{T}}\right) + \mathcal{R}_T\left(0, \frac{a}{\sqrt{T}}\right) \right) = 0.$$

Therefore, we infer from (2.3), (2.4) and (2.5) that

$$(2.6) \quad \lim_{T \rightarrow \infty} \Lambda_T(a) = \exp\left(-\frac{a^2}{4\theta^3}\right).$$

Convergence (2.6) holds for all  $a$  in a neighborhood of the origin which leads to (2.2) and completes the proof of Proposition 2.  $\square$

In order to obtain the large deviation properties for  $(S_T)$ , we shall make use of Lemma 1 with  $a = 0$  and  $b = a$ . On the one hand, let

$$D_H = \left\{ a \in \mathbb{R} / \theta^2 - 2a > 0 \text{ and } \sqrt{\theta^2 - 2a} > -\delta_H \theta \right\}.$$

It is not hard to see that  $D_H = ]-\infty, a_H[$  where

$$(2.7) \quad a_H = \frac{\theta^2}{2}(1 - \delta_H^2).$$

Consequently, as  $|\delta_H| < 1$ , one can observe that the origin always belongs to the interior of  $D_H$ . On the other hand, for all  $a \in D_H$ , let  $\varphi(a) = \sqrt{\theta^2 - 2a}$ ,

$$(2.8) \quad L(a) = \mathcal{L}(0, a) = -\frac{1}{2}(\theta + \sqrt{\theta^2 - 2a}),$$

$$(2.9) \quad H(a) = \mathcal{H}(0, a) = -\frac{1}{2} \log\left(\frac{\varphi(a) - \theta}{2\varphi(a)}\right),$$

$$(2.10) \quad K_T(a) = \mathcal{K}_T(0, a) = -\frac{1}{2} \log\left(1 + \frac{(\varphi(a) + \theta)}{2\varphi(a)} r_T(a)\right).$$

The main difficulty comparing to [3] is that the function  $L$  is not steep. Actually,

$$L'(a) = \frac{1}{2\sqrt{\theta^2 - 2a}}$$

which implies that  $L'(a_H) = -1/(2\theta\delta_H)$ . Moreover, for all  $c > 0$ ,  $L'(a) = c$  if and only if  $a = a_c$  with  $a_c = (4\theta^2c^2 - 1)/(8c^2)$ . Hence,  $a_c < a_H$  whenever  $0 < c < -1/(2\theta\delta_H)$ . Denote by  $I$  the Fenchel-Legendre transform of the function  $L$ . Our first result on the large deviation properties of  $(S_T/T)$  is as follows.

**Theorem 3.** *The sequence  $(S_T/T)$  satisfies an LDP with good rate function*

$$(2.11) \quad I(c) = \begin{cases} \frac{(2\theta c + 1)^2}{8c} & \text{if } 0 < c \leq -\frac{1}{2\theta\delta_H}, \\ \frac{c\theta^2}{2}(1 - \delta_H^2) + \frac{\theta}{2}(1 - \delta_H) & \text{if } c \geq -\frac{1}{2\theta\delta_H}, \\ +\infty & \text{otherwise.} \end{cases}$$

**Remark 2.** *In the particular case  $H = 1/2$ , then  $\delta_H = 0$  and the LDP for  $(S_T/T)$  is exactly the one established by Bryc and Dembo [6] for general centered Gaussian processes.*

We are now going to improve this result by a Sharp Large Deviation Principle (SLDP) for  $(S_T/T)$  inspired by the well-known Bahadur-Rao Theorem [1] on the sample mean.

**Theorem 4.** *The sequence  $(S_T/T)$  satisfies a SLDP associated with  $L$ ,  $H$  and  $K_T$  given by (2.8), (2.9), and (2.10), respectively.*

a) *For all  $-1/(2\theta) < c < -1/(2\theta\delta_H)$ , there exists a sequence  $(b_{c,k}^H)$  such that, for any  $p > 0$  and  $T$  large enough,*

$$(2.12) \quad \mathbb{P}(S_T \geq cT) = \frac{\exp(-TI(c) + J(c) + K_H(c))}{a_c \sigma_c \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^p \frac{b_{c,k}^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \right]$$

while, for  $0 < c < -1/(2\theta)$ ,

$$(2.13) \quad \mathbb{P}(S_T \leq cT) = -\frac{\exp(-TI(c) + J(c) + K_H(c))}{a_c \sigma_c \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^p \frac{b_{c,k}^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \right]$$

where

$$(2.14) \quad J(c) = -\frac{1}{2} \log \left( \frac{1 - 2\theta c}{2} \right), \quad K_H(c) = -\frac{1}{2} \log \left( \frac{(1 + \sin(\pi H))(1 + 2\theta c\delta_H)}{2 \sin(\pi H)} \right),$$

with

$$(2.15) \quad a_c = \frac{4\theta^2 c^2 - 1}{8c^2} \quad \text{and} \quad \sigma_c^2 = 4c^3.$$

Moreover, the coefficients  $b_{c,1}^H, b_{c,2}^H, \dots, b_{c,p}^H$  may be explicitly calculated as functions of the derivatives of  $L$  and  $H$  evaluated at point  $a_c$ . They also depend on the Taylor expansion of  $K_T$  and its derivatives at  $a_c$ .

b) *For all  $c > -1/(2\theta\delta_H)$ , there exists a sequence  $(d_{c,k}^H)$  such that, for any  $p > 0$  and  $T$  large enough*

$$(2.16) \quad \mathbb{P}(S_T \geq cT) = \frac{\exp(-TI(c) + P_H(c) + Q_H(c))}{a_H \sigma_H \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^p \frac{d_{c,k}^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \right]$$

$$(2.17) \quad P_H(c) = -\frac{1}{2} \log \left( \frac{-(1+2\theta c \delta_H)}{4\delta_H \sin(\pi H)} \right), \quad Q_H(c) = \frac{(2H-1)^2 \sin(\pi H)(1+2\theta c \delta_H)}{2(1-(\sin(\pi H))^2)}$$

with

$$(2.18) \quad a_H = \frac{\theta^2(1-\delta_H^2)}{2} \quad \text{and} \quad \sigma_H^2 = -\frac{1}{2\theta^3\delta_H^3}.$$

Moreover, the coefficients  $d_{c,1}^H, d_{c,2}^H, \dots, d_{c,p}^H$  may be explicitly calculated as functions of the derivatives of  $L$  and  $H$  evaluated at point  $a_H$ . They also depend on the Taylor expansion of  $K_T$  and its derivatives at  $a_H$ .

**Remark 3.** For example, the first coefficient  $b_{c,1}^H$  is given by

$$b_{c,1}^H = \frac{1}{\sigma_c^2} \left( \frac{s_1}{a_c} - \frac{s_1^2}{2} - \frac{s_2}{2} - \frac{s_3}{2a_c\sigma_c^2} + \frac{s_1\ell_3}{2\sigma_c^2} - \frac{5\ell_3^2}{24\sigma_c^4} + \frac{\ell_4}{8\sigma_c^2} - \frac{1}{a_c^2} \right) + k_{c,1}^H$$

where  $\ell_q = L^{(q)}(a_c)$ ,  $h_q = H^{(q)}(a_c)$ ,  $s_q = h_q + k_q$ ,  $p_H = (1 - \sin(\pi H))/\sin(\pi H)$ , with

$$\begin{aligned} k_1 &= \lim_{T \rightarrow \infty} K'_T(a_c) = \frac{-4\theta p_H c^3}{2 + p_H(1 + 2\theta c)}, \\ k_2 &= \lim_{T \rightarrow \infty} K''_T(a_c) = \frac{16\theta p_H c^5(6 + p_H(3 + 2\theta c))}{(2 + p_H(1 + 2\theta c))^2}, \\ k_{c,1}^H &= \lim_{T \rightarrow \infty} T(K_T(a_c) - K_H(c)) = \frac{c(1 + 2\theta c)(2H - 1)^2}{2 \sin(\pi H)(2 + p_H(1 + 2\theta c))}. \end{aligned}$$

In addition, one can also observe that  $\sigma_c^2 = L''(a_c)$  and  $\sigma_H^2 = L''(a_H)$ .

**Theorem 5** For  $c = -1/(2\theta\delta_H)$ , there exists a sequence  $(d_k^H)$  such that, for any  $p > 0$  and  $T$  large enough

$$(2.19) \quad \mathbb{P}(S_T \geq cT) = \frac{\exp(-TI(c) + K_H)\Gamma(1/4)}{2\pi a_H \sigma_H T^{1/4}} \left[ 1 + \sum_{k=1}^{2p} \frac{d_k^H}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p \sqrt{T}}\right) \right]$$

where  $a_H$  and  $\sigma_H^2$  are given by (2.18) and

$$K_H = \frac{1}{2} \log(\delta_H \sin(\pi H)) + \frac{1}{4} \log(-\theta \delta_H).$$

As before, the coefficients  $d_1^H, d_2^H, \dots, d_{2p}^H$  may be explicitly calculated.

**Proof.** The proofs are given in Section 4. □

### 3 The maximum likelihood estimator.

Our purpose is now to establish similar results for the maximum likelihood estimator. The almost sure convergence of  $\hat{\theta}_T$  towards  $\theta$  was already proven by Kleptsyna and Le Breton [10], see also Prakasa Rao [13, 14] for related results. We just learn that the central limit theorem was established via a different approach by Brouste and Kleptsyna [5].

**Proposition 6.** *We have*

$$(3.1) \quad \lim_{T \rightarrow \infty} \widehat{\theta}_T = \theta \quad a.s.$$

Moreover, we also have the CLT

$$(3.2) \quad \sqrt{T}(\widehat{\theta}_T - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, -2\theta).$$

**Proof of Proposition 6.** The almost sure convergence (3.1) is given by Proposition 2.2 of Kleptsyna and Le Breton [10]. For all  $c \in \mathbb{R}$ , denote

$$V_T(c) = \frac{1}{\sqrt{T}} \int_0^T Q_t dY_t - \left( \frac{c}{\sqrt{T}} + \theta \right) \frac{S_T}{\sqrt{T}}.$$

One can easily check that

$$(3.3) \quad \mathbb{P}(\sqrt{T}(\widehat{\theta}_T - \theta) \leq c) = \mathbb{P}(V_T(c) \leq 0).$$

Consequently, in order to prove the CLT given by (3.2), it is only necessary to establish the asymptotic behavior the sequence  $(V_T(c))$  where the threshold  $c$  can be seen as a parameter. For all  $a, c \in \mathbb{R}$ , let  $\Lambda_T(a, c) = \mathbb{E}[\exp(aV_T(c))]$ . We clearly have

$$\Lambda_T(a, c) = \exp \left( T L_T \left( \frac{a}{\sqrt{T}}, c_T \right) \right) \quad \text{where} \quad c_T = -\frac{a}{\sqrt{T}} \left( \frac{c}{\sqrt{T}} + \theta \right).$$

Thus, it follows from the decomposition (1.9) that

$$(3.4) \quad \Lambda_T(a, c) = \exp \left( T \mathcal{L} \left( \frac{a}{\sqrt{T}}, c_T \right) + \mathcal{H} \left( \frac{a}{\sqrt{T}}, c_T \right) + \mathcal{K}_T \left( \frac{a}{\sqrt{T}}, c_T \right) + \mathcal{R}_T \left( \frac{a}{\sqrt{T}}, c_T \right) \right).$$

On the one hand,

$$\mathcal{L} \left( \frac{a}{\sqrt{T}}, c_T \right) = -\frac{1}{2} \left( \frac{a}{\sqrt{T}} + \theta + \varphi_T \right)$$

where

$$\varphi_T = \sqrt{\theta^2 - 2c_T} = -\theta \sqrt{1 - \frac{2c_T}{\theta^2}}.$$

Hence, as

$$\varphi_T = -\theta - \frac{a}{\sqrt{T}} + \frac{(a^2 - 2ac)}{2\theta T} + o \left( \frac{1}{T} \right)$$

we deduce that

$$(3.5) \quad \lim_{T \rightarrow \infty} T \mathcal{L} \left( \frac{a}{\sqrt{T}}, c_T \right) = -\frac{(a^2 - 2ac)}{4\theta}.$$

On the other hand, as  $(\varphi_T)$  converges to  $-\theta$ , it is not hard to see that

$$(3.6) \quad \lim_{T \rightarrow \infty} \left( \mathcal{H} \left( \frac{a}{\sqrt{T}}, c_T \right) + \mathcal{K}_T \left( \frac{a}{\sqrt{T}}, c_T \right) + \mathcal{R}_T \left( \frac{a}{\sqrt{T}}, c_T \right) \right) = 0.$$

The conjunction of (3.4), (3.5) and (3.6) leads to

$$(3.7) \quad \lim_{T \rightarrow \infty} \Lambda_T(a, c) = \exp\left(-\frac{(a^2 - 2ac)}{4\theta}\right).$$

This convergence holds for all  $a$  in a neighborhood of the origin. Consequently,

$$(3.8) \quad V_T(c) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\frac{c}{2\theta}, \frac{-1}{2\theta}\right).$$

Denote by  $V(c)$  the limiting distribution of  $(V_T(c))$ . It follows from a standard Gaussian calculation that

$$(3.9) \quad \mathbb{P}(V(c) \leq 0) = \frac{1}{-4\pi\theta} \int_{-\infty}^c \exp\left(\frac{x^2}{2\theta}\right) dx.$$

Finally, (3.3) and (3.9) imply (3.2) which achieves the proof of Proposition 6.  $\square$

In order to establish the large deviation properties of  $(\hat{\theta}_T)$ , we shall make use of the auxiliary random variable defined for all  $c \in \mathbb{R}$  by

$$(3.10) \quad Z_T(c) = \int_0^T Q_t dY_t - c \int_0^T Q_t^2 d\langle M \rangle_t$$

since  $\mathbb{P}(\hat{\theta}_T \geq c) = \mathbb{P}(Z_T(c) \geq 0)$ . Let

$$D_H = \left\{ a \in \mathbb{R} / \theta^2 + 2ac > 0 \text{ and } \sqrt{\theta^2 + 2ac} > \max(a + \theta; -\delta_H(a + \theta)) \right\}.$$

After some straightforward calculations, it is not hard to see that

$$D_H = \begin{cases} ]a_1^H, a_2^H[ & \text{if } c \leq \frac{\theta}{2}, \\ ]a_1^H, a^c[ & \text{if } c > \frac{\theta}{2}, \end{cases}$$

where  $a^c = 2(c - \theta)$  and

$$\begin{cases} a_1^H = \frac{c - \theta\mu_H - \sqrt{c^2 - 2\theta c\mu_H + \theta^2\mu_H}}{\mu_H}, \\ a_2^H = \frac{c - \theta\mu_H + \sqrt{c^2 - 2\theta c\mu_H + \theta^2\mu_H}}{\mu_H}, \end{cases}$$

where  $\mu_H = \delta_H^2$ . In addition, for all  $a \in D_H$ , let  $\varphi(a) = \sqrt{\theta^2 + 2ac}$  and

$$(3.11) \quad L(a) = \mathcal{L}(a, -ca) = -\frac{1}{2} \left( a + \theta + \sqrt{\theta^2 + 2ac} \right),$$

$$(3.12) \quad H(a) = \mathcal{H}(a, -ca) = -\frac{1}{2} \log\left(\frac{\varphi(a) - a - \theta}{2\varphi(a)}\right),$$

$$(3.13) \quad K_T(a) = \mathcal{K}_T(a, -ca) = -\frac{1}{2} \log\left(1 + \frac{(\varphi(a) + a + \theta)}{2\varphi(a)} r_T(-ac)\right).$$

The function  $L$  is not steep as the derivative of  $L$  is finite at the boundary of  $D_H$ . Moreover,  $L'(a) = 0$  if and only if  $a = a_c$  with  $a_c = (c^2 - \theta^2)/(2c)$  and  $a_c \in D_H$  whenever  $c < \theta/3$ .

**Theorem 7.** *The maximum likelihood estimator  $(\hat{\theta}_T)$  satisfies an LDP with good rate function*

$$(3.14) \quad I(c) = \begin{cases} -\frac{(c-\theta)^2}{4c} & \text{if } c < \frac{\theta}{3}, \\ 2c - \theta & \text{if } c \geq \frac{\theta}{3}. \end{cases}$$

**Remark 4.** *One can observe that the rate function  $I$  is totally free of the parameter  $H$ . Hence,  $(\hat{\theta}_T)$  shares the same LDP than the one established by Florens-Landais and Pham [8] for the standard Ornstein-Uhlenbeck process with  $H = 1/2$ .*

**Theorem 8.** *The maximum likelihood estimator  $(\hat{\theta}_T)$  satisfies an SLDP associated with  $L$ ,  $H$  and  $K_T$  given by (3.11), (3.12), and (3.13), respectively.*

a) *For all  $\theta < c < \theta/3$ , there exists a sequence  $(b_{c,k}^H)$  such that, for any  $p > 0$  and  $T$  large enough,*

$$(3.15) \quad \mathbb{P}(\hat{\theta}_T \geq c) = \frac{\exp(-TI(c) + J(c) + K_H(c))}{\sigma_c a_c \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^p \frac{b_{c,k}^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \right]$$

while, for  $c < \theta$ ,

$$(3.16) \quad \mathbb{P}(\hat{\theta}_T \leq c) = -\frac{\exp(-TI(c) + J(c) + K_H(c))}{\sigma_c a_c \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^p \frac{b_{c,k}^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \right]$$

where

$$(3.17) \quad J(c) = -\frac{1}{2} \log \left( \frac{(c+\theta)(3c-\theta)}{4c^2} \right), \quad K_H(c) = -\frac{1}{2} \log \left( 1 + p_H \frac{(c-\theta)^2}{4c^2} \right)$$

with  $p_H = (1 - \sin(\pi H))/\sin(\pi H)$ ,

$$(3.18) \quad a_c = \frac{c^2 - \theta^2}{2c} \quad \text{and} \quad \sigma_c^2 = -\frac{1}{2c}.$$

Moreover, the coefficients  $b_{c,1}^H, b_{c,2}^H, \dots, b_{c,p}^H$  may be explicitly calculated as in Theorem 4.

b) *For all  $c > \theta/3$  with  $c \neq 0$ , there exists a sequence  $(d_{c,k}^H)$  such that, for any  $p > 0$  and  $T$  large enough,*

$$(3.19) \quad \mathbb{P}(\hat{\theta}_T \geq c) = \frac{\exp(-TI(c) + P(c)) \sqrt{\sin(\pi H)}}{\sigma_c a_c \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^p \frac{d_{c,k}^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \right]$$

where

$$(3.20) \quad P(c) = -\frac{1}{2} \log \left( \frac{(c-\theta)(3c-\theta)}{4c^2} \right)$$

with

$$(3.21) \quad a_c = 2(c - \theta) \quad \text{and} \quad (\sigma_c^2)^2 = \frac{c^2}{2(2c - \theta)^3}.$$

In addition, the coefficients  $d_{c,1}^H, d_{c,2}^H, \dots, d_{c,p}^H$  may be explicitly calculated as in Theorem 4.  
c) Finally, for  $c = 0$ , for any  $p > 0$  and for  $T$  large enough,

$$(3.22) \quad \mathbb{P}(\hat{\theta}_T \geq 0) = 2 \frac{\exp(-TI(c))\sqrt{\sin(\pi H)}}{\sqrt{2\pi T}\sqrt{-2\theta}} \left[ 1 + \sum_{k=1}^p \frac{d_k^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \right]$$

As before, the coefficients  $d_1^H, d_2^H, \dots, d_p^H$  may be explicitly calculated.

**Theorem 9** For  $c = \theta/3$ , there exists a sequence  $(d_k^H)$  such that, for any  $p > 0$  and  $T$  large enough

$$(3.23) \quad \mathbb{P}(\hat{\theta}_T \geq \frac{\theta}{3}) = \frac{\exp(-TI(c))\Gamma(\frac{1}{4})}{4\pi T^{1/4} a_\theta^{3/4} \sigma_\theta} \sqrt{\sin(\pi H)} \left[ 1 + \sum_{k=1}^p \frac{e_k^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \right]$$

where  $a_\theta$  and  $\sigma_\theta$  are given by

$$(3.24) \quad a_\theta = -\frac{4\theta}{3} \quad \text{and} \quad \sigma_\theta^2 = -\frac{3}{2\theta}.$$

As before, the coefficients  $e_1^H, e_2^H, \dots, e_p^H$  may be explicitly calculated.

**Proof.** The proofs are left to the reader as they follow essentially the same lines as that for the energy.  $\square$

## 4 Proofs of the main results.

### 4.1 Proof of Theorem 4, first part.

We first focus our attention on the sharp LDP for the energy  $S_T$  in the easy case  $-1/(2\theta) < c < -1/(2\theta\delta_H)$ . Let  $L_T$  be the normalized cumulant generating function of  $S_T$ . We already saw that  $a_c$ , given by (2.11), belongs to the domain  $D_H$  whenever  $c < -1/(2\theta\delta_H)$ . For all  $-1/(2\theta) < c < -1/(2\theta\delta_H)$ , we can split  $\mathbb{P}(S_T \geq cT)$  into two terms,  $\mathbb{P}(S_T \geq cT) = A_T B_T$  with

$$(4.25) \quad A_T = \exp(T(L_T(a_c) - ca_c)),$$

$$(4.26) \quad B_T = \mathbb{E}_T \left[ \exp(-a_c(S_T - cT)) \mathbb{I}_{S_T \geq cT} \right],$$

where  $\mathbb{E}_T$  stands for the expectation after the usual change of probability

$$(4.27) \quad \frac{d\mathbb{P}_T}{d\mathbb{P}} = \exp\left(a_c S_T - T L_T(a_c)\right).$$

On the one hand, we can deduce from (1.9) with  $L(a) = \mathcal{L}(0, a)$ ,  $H(a) = \mathcal{H}(0, a)$ ,  $K_T(a) = \mathcal{K}_T(0, a)$ , and  $R_T(a) = \mathcal{R}_T(0, a)$  together with (2.11) and (2.14) that

$$(4.28) \quad \begin{aligned} A_T &= \exp\left(T(L(a_c) - ca_c) + H(a_c) + K_T(a_c) + R_T(a_c)\right), \\ A_T &= \exp\left(-TI(c) + J(c) + K_T(a_c) + R_T(a_c)\right). \end{aligned}$$

Consequently, we infer from (2.10) and (2.14) that for any  $p > 0$  and  $T$  large enough

$$(4.29) \quad K_T(a_c) = K_H(c) + \sum_{k=1}^p \frac{\gamma_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)$$

where the coefficients  $(\gamma_k)$ , which also depend on  $H$ , may be explicitly calculated. In addition, it is not hard to see from (1.13) that the remainder  $R_T(a_c) = \mathcal{O}(\exp(-T/c))$ . Therefore, we deduce from (4.28) and (4.29) that for any  $p > 0$  and  $T$  large enough

$$(4.30) \quad A_T = \exp(-TI(c) + J(c) + K_H(c)) \left[ 1 + \sum_{k=1}^p \frac{\alpha_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \right]$$

where, as before, the coefficients  $(\alpha_k)$  may be explicitly calculated. For example,

$$\alpha_1 = \frac{-2c(1+2\theta c)r_1^H}{\sin(\pi H)(2+(1+2\theta c)p_H)}.$$

It now remains to give the expansion for  $B_T$  which can be rewritten as

$$(4.31) \quad B_T = \mathbb{E}_T \left[ \exp(-a_c \sigma_c \sqrt{T} U_T) \mathbb{I}_{U_T \geq 0} \right]$$

where

$$U_T = \frac{S_T - cT}{\sigma_c \sqrt{T}}.$$

**Lemma 10.** *For all  $-1/(2\theta) < c < -1/(2\theta\delta_H)$ , the distribution of  $U_T$  under  $\mathbb{P}_T$  converges, as  $T$  goes to infinity, to the  $\mathcal{N}(0, 1)$  distribution. Moreover, there exists a sequence  $(\beta_k)$  such that, for any  $p > 0$  and  $T$  large enough,*

$$(4.32) \quad B_T = \frac{\beta_0}{\sqrt{T}} \left[ 1 + \sum_{k=1}^p \frac{\beta_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \right].$$

The sequence  $(\beta_k)$  only depends on the derivatives of  $L$  and  $H$  evaluated at point  $a_c$ . They also depend on the Taylor expansion of  $K_T$  and its derivatives at  $a_c$ . For example,

$$\begin{aligned} \beta_0 &= \frac{1}{a_c \sigma_c \sqrt{2\pi}}, \\ \beta_1 &= \frac{1}{\sigma_c^2} \left( \frac{s_1}{a_c} - \frac{s_1^2}{2} - \frac{s_2}{2} - \frac{s_3}{2a_c \sigma_c^2} + \frac{s_1 \ell_3}{2\sigma_c^2} - \frac{5\ell_3^2}{24\sigma_c^4} + \frac{\ell_4}{8\sigma_c^2} - \frac{1}{a_c^2} \right), \end{aligned}$$

where  $\ell_q = L^{(q)}(a_c)$ ,  $h_q = H^{(q)}(a_c)$ , and  $s_q = h_q + k_q$ .

**Proof.** The proof of Lemma 10 is given in Appendix B. □

**Proof of Theorem 4, first part.** The expansions (2.12) and ((2.13) immediately follow from the conjunction of (4.30) and (4.32). □

## 4.2 Proof of Theorem 4, second part.

We are now going to establish the sharp LDP for the energy  $S_T$  in the more complicated case  $c > -1/(2\theta\delta_H)$ . We have already seen that the point  $a_c$ , given by (2.11), belongs to the domain  $D_H = ]-\infty, a_H[$  whenever  $c < -1/(2\theta\delta_H)$ . Consequently, for  $c > -1/(2\theta\delta_H)$ , it is necessary to make use of a slight modification of the strategy of time varying change of probability proposed by Bryc and Dembo [6].

Now, we show a modification of the decomposition (1.9) which allows us to replace  $r_T(a)$  by  $p_H$  in  $K_T$ , putting the difference into the remainder term. The modified remainder will be denoted by  $\check{R}_T$ .

**Lemma 11.** *For any  $a \in D_H$ , we have the following decomposition*

$$(4.33) \quad L_T(a) = L(a) + \frac{1}{T}H(a) + \frac{1}{T}K(a) + \frac{1}{T}\check{R}_T(a)$$

where  $L$  and  $H$  are given by (2.8) and (2.9) respectively,

$$(4.34) \quad K(a) = -\frac{1}{2} \log \left( 1 + \frac{\varphi(a) + \theta}{2\varphi(a)} p_H \right),$$

and the remainder term

$$(4.35) \quad \begin{aligned} \check{R}_T(a) = & -\frac{1}{2} \log \left( 1 + \frac{(\varphi(a) + \theta)(r_T(a) - p_H)}{(2 + p_H)\varphi(a) + \theta\delta_H} \right. \\ & \left. + \frac{(\varphi(a) + \theta)^2}{(\varphi(a) - \theta)((2 + p_H)\varphi(a) + \theta\delta_H)} e^{-2T\varphi(a)} \right). \end{aligned}$$

**Proof of Lemma 11.** It follows from (1.9) that

$$\begin{aligned} L_T(a) &= L(a) + \frac{1}{T}H(a) + \frac{1}{T}K_T(a) + \frac{1}{T}R_T(a), \\ &= L(a) + \frac{1}{T}H(a) + \frac{1}{T}K(a) + \frac{1}{T}\check{R}_T(a) \end{aligned}$$

where

$$\check{R}_T(a) = K_T(a) - K(a) + R_T(a).$$

Hence, denote  $\varphi = \sqrt{\theta^2 - 2a}$ , it is not hard to see that

$$\begin{aligned} \exp(-2\check{R}_T(a)) &= \left( \frac{2\varphi + (\varphi + \theta)r_T(a)}{2\varphi + (\varphi + \theta)p_H} \right) \left( 1 + \frac{(\varphi + \theta)^2}{(\varphi - \theta)(2\varphi + (\varphi + \theta)r_T(a))} e^{-2T\varphi} \right), \\ &= \frac{2\varphi + (\varphi + \theta)r_T(a)}{2\varphi + (\varphi + \theta)p_H} + \frac{(\varphi + \theta)^2}{(\varphi - \theta)(2\varphi + (\varphi + \theta)p_H)} e^{-2T\varphi}, \\ &= 1 + \frac{(\varphi + \theta)(r_T(a) - p_H)}{2\varphi + (\varphi + \theta)p_H} + \frac{(\varphi + \theta)^2}{(\varphi - \theta)(2\varphi + (\varphi + \theta)p_H)} e^{-2T\varphi}. \end{aligned}$$

which ends the proof of Lemma 11. □

Denote by  $\Lambda_T$  the suitable approximation of the normalized cumulant generating function  $L_T$  given by

$$(4.36) \quad \Lambda_T(a) = L(a) + \frac{1}{T}H(a) + \frac{1}{T}K(a)$$

The previous lemma enable us to write

$$(4.37) \quad L_T(a) = \Lambda_T(a) + \frac{1}{T}\check{R}_T(a).$$

One can observe that  $\Lambda_T$  is an holomorph function in the domain  $D_H$ . In addition, there exists a unique  $a_T$ , which belongs to the interior of  $D_H$  and converges to its border  $a_H$ , solution of the implicite equation

$$(4.38) \quad \Lambda'_T(a) = c.$$

After some tedious but straightforward calculations, we can deduce from (4.38) that there exists a sequence  $(a_k)$  such that, for any  $p > 0$  and  $T$  large enough,

$$(4.39) \quad a_T = \sum_{k=0}^p \frac{a_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)$$

with

$$a_0 = a_H, \quad a_1 = -\frac{\theta\delta_H}{1 + 2\theta c\delta_H},$$

$$a_2 = \frac{(2\theta c\delta_H(4 + \sin(\pi H)) + 2 + \sin(\pi H))}{2(1 + 2\theta c\delta_H)^3}.$$

Moreover, if  $\varphi_T = \varphi(a_T) = \sqrt{\theta^2 - 2a_T}$ , we also have for any  $p > 0$  and  $T$  large enough,

$$(4.40) \quad \varphi_T = \sum_{k=0}^p \frac{\varphi_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)$$

with

$$\varphi_0 = -\theta\delta_H, \quad \varphi_1 = \frac{-1}{1 + 2\theta c\delta_H},$$

$$\varphi_2 = \frac{(2\theta c\delta_H(5 + \sin(\pi H)) + 3 + \sin(\pi H))}{2\theta\delta_H(1 + 2\theta c\delta_H)^3}.$$

Hereafter, we introduce the new probability measure

$$(4.41) \quad \frac{d\mathbb{P}_T}{d\mathbb{P}} = \exp\left(a_TS_T - TL_T(a_T)\right)$$

and we denote by  $\mathbb{E}_T$  the expectation under  $\mathbb{P}_T$ . It clearly leads to the decomposition  $\mathbb{P}(S_T \geq cT) = A_T B_T$  where

$$(4.42) \quad A_T = \exp(TL_T(a_T) - cTa_T),$$

$$(4.43) \quad B_T = \mathbb{E}_T\left[\exp(-a_T(S_T - cT))\mathbb{I}_{S_T \geq cT}\right].$$

The proof now splits into two parts, the first one is devoted to the expansion of  $A_T$  while the second one gives the expansion of  $B_T$ . It follows from (4.36), (4.37) and (4.42) that

$$(4.44) \quad A_T = \exp\left(T(L(a_T) - ca_T) + H(a_T) + K(a_T) + \check{R}_T(a_T)\right).$$

We can deduce from the Taylor expansions of  $a_T$  and  $\varphi_T$  given by (4.39) and (4.40) that

$$\begin{aligned} T(L(a_T) - ca_T) &= -\frac{T}{2}(\theta + \varphi_T + 2ca_T), \\ &= -T(ca_H - L(a_H)) - \frac{\varphi_1}{2} - ca_1 - \frac{1}{2} \sum_{k=1}^p \frac{\varphi_{k+1} + 2ca_{k+1}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right), \\ &= -TI(c) + \frac{1}{2} - \frac{1}{2} \sum_{k=1}^p \frac{\varphi_{k+1} + 2ca_{k+1}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right). \end{aligned}$$

Consequently, we obtain that for any  $p > 0$  and  $T$  large enough,

$$(4.45) \quad \exp\left(T(L(a_T) - ca_T)\right) = \exp(-TI(c))\sqrt{e} \left[1 + \sum_{k=1}^p \frac{\alpha_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]$$

where the coefficients  $(\alpha_k)$  may be explicitly calculated. For example,

$$\alpha_1 = \frac{-1}{4\theta\delta_H(1 + 2c\theta\delta_H)^2} (2\theta c\delta_H(4 + \sin(\pi H)) + 3 + \sin(\pi H)).$$

By the same way, we find that for any  $p > 0$  and  $T$  large enough,

$$(4.46) \quad \exp(H(a_T)) = \sqrt{\frac{2\varphi_T}{\varphi_T - \theta}} = \sqrt{1 - \sin(\pi H)} \left[1 + \sum_{k=1}^p \frac{\beta_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]$$

where the coefficients  $(\beta_k)$  may be explicitly calculated. For example,

$$\beta_1 = \frac{1 + \sin(\pi H)}{4\theta\delta_H(1 + 2\theta c\delta_H)}.$$

The expansions for  $K(a_T)$  and  $\check{R}_T(a_T)$  are much more tricky. On the one hand,

$$\exp(K(a_T)) = \sqrt{\frac{2\varphi_T}{2\varphi_T + (\varphi_T + \theta)p_H}}.$$

One can observe that  $2\varphi_0 + (\varphi_0 + \theta)p_H = 0$ . Hence, multiplying the numerator and the denominator by  $T$ , we obtain that for any  $p > 0$  and  $T$  large enough,

$$\begin{aligned} \exp(K(a_T)) &= \sqrt{\frac{2T\varphi_T}{2T\varphi_T + T(\varphi_T + \theta)p_H}}, \\ (4.47) \quad &= \sqrt{\theta T \delta_H (1 - \delta_H) (1 + 2\theta c\delta_H)} \left[1 + \sum_{k=1}^p \frac{\gamma_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right] \end{aligned}$$

where, as before, the coefficients  $(\gamma_k)$  may be explicitly calculated. On the other hand, the remainder  $\check{R}_T(a_T) = K_T(a_T) - K(a_T) + R_T(a_T)$ . It is not hard to see that

$$\begin{aligned}\exp(-2\check{R}_T(a_T)) &= \frac{2\varphi_T + (\varphi_T + \theta)r_T(a_T)}{2\varphi_T + (\varphi_T + \theta)p_H} + \frac{(\varphi_T + \theta)^2}{(\varphi_T - \theta)(2\varphi_T + (\varphi_T + \theta)p_H)} \exp(-2T\varphi_T), \\ &= \frac{2\varphi_T + (\varphi_T + \theta)r_T(a_T)}{2\varphi_T + (\varphi_T + \theta)p_H} + \mathcal{O}(T \exp(2\theta T \delta_H)).\end{aligned}$$

Therefore,

$$\exp(\check{R}_T(a_T)) = \sqrt{\frac{2\varphi_T + (\varphi_T + \theta)p_H}{2\varphi_T + (\varphi_T + \theta)r_T(a_T)}} \left(1 + \mathcal{O}(T \exp(2\theta T \delta_H))\right).$$

Recall that  $r_T(a) = r_H(\varphi(a)T/2) \exp(-T\varphi(a)) - 1$ . It is shown (equation (A.9) in Appendix A) that for any  $p > 0$  and  $T$  large enough,

$$(4.48) \quad r_T(a_T) = p_H + \frac{1}{\sin \pi H} \sum_{k=1}^p \frac{2^k r_k^H}{\varphi_T^k T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)$$

where the coefficients  $(r_k^H)$  may be explicitly calculated. For example,

$$r_1^H = -\frac{(2H-1)^2}{4}.$$

Consequently, we infer from (4.48) that

$$T(r_T(a_T) - p_H) = w_T(a_T) + \mathcal{O}\left(\frac{1}{T^p}\right)$$

where

$$w_T(a_T) = \frac{1}{\sin(\pi H)} \sum_{k=1}^p \frac{2^k r_k^H}{\varphi_T^k T^{k-1}}.$$

If  $\mu_T = T(2\varphi_T + (\varphi_T + \theta)p_H)$ , we obtain that for any  $p > 0$  and  $T$  large enough,

$$\begin{aligned}\exp(\check{R}_T(a_T)) &= \sqrt{\frac{\mu_T}{\mu_T + (\varphi_T + \theta)T(r_T(a_T) - p_H)}} \left(1 + \mathcal{O}(T \exp(2\theta T \delta_H))\right), \\ (4.49) \quad &= \sqrt{\frac{1 - (\sin(\pi H))^2}{1 - (\sin(\pi H))^2 + 4r_1^H(\sin(\pi H))(1 + 2\theta c \delta_H)}} \left[1 + \sum_{k=1}^p \frac{\delta_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]\end{aligned}$$

where, as before, the coefficients  $(\delta_k)$  may be explicitly calculated. Putting together the four contributions (4.45), (4.46), (4.47), and (4.49), we find from (4.44) that for any  $p > 0$  and  $T$  large enough,

$$(4.50) \quad A_T = \exp(-TI(c) + R_H(c)) \delta_H \sqrt{2e\theta T \sin(\pi H)(1 + 2\theta c \delta_H)} \left[1 + \sum_{k=1}^p \frac{\alpha_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]$$

where the coefficients  $(\alpha_k)$  may be explicitly calculated and

$$(4.51) \quad R_H(c) = -\frac{1}{2} \log \left(1 - \frac{(2H-1)^2 \sin(\pi H)(1 + 2\theta c \delta_H)}{1 - (\sin(\pi H))^2}\right).$$

The rest of the proof concerns the expansion of  $B_T$  which can be rewritten as

$$(4.52) \quad B_T = \mathbb{E}_T \left[ \exp(-a_T T U_T) \mathbb{I}_{U_T \geq 0} \right]$$

where

$$U_T = \frac{S_T - cT}{T}.$$

**Lemma 12.** *For all  $c > -1/(2\theta\delta_H)$ , the distribution of  $U_T$  under  $\mathbb{P}_T$  converges, as  $T$  goes to infinity, to the distribution of  $\nu_H N^2 - \gamma_H$  where  $N$  stands for the standard  $\mathcal{N}(0, 1)$  distribution,*

$$(4.53) \quad \gamma_H = c - L'(a_H) = \frac{1 + 2\theta c \delta_H}{2\theta \delta_H},$$

$$(4.54) \quad \nu_H = \frac{(1 - (\sin(\pi H))^2)\gamma_H}{1 - (\sin(\pi H))^2 - (2H - 1)^2 \sin(\pi H)(1 + 2\theta c \delta_H)}.$$

In other words, the limit of the characteristic function of  $U_T$  under  $\mathbb{E}_T$  is

$$(4.55) \quad \Phi(u) = \frac{\exp(-i\gamma_H u)}{\sqrt{1 - 2i\nu_H u}}.$$

Moreover, there exists a sequence  $(\beta_k)$  such that, for any  $p > 0$  and  $T$  large enough,

$$(4.56) \quad B_T = \sum_{k=1}^p \frac{\beta_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right).$$

The sequence  $(\beta_k)$  only depends on the Taylor expansion of  $a_T$  at the neighborhood of  $a_H$  together with the derivatives of  $L$  and  $H$  evaluated at point  $a_H$ . They also depend on the Taylor expansion of  $K_T$  and its derivatives at  $a_H$ . For example,

$$\beta_1 = \frac{1}{a_H \gamma_H \sqrt{2\pi e}} \exp(Q_H(c) - R_H(c))$$

where  $R_H(c)$  is given by (4.51) and

$$Q_H(c) = \frac{(2H - 1)^2 \sin(\pi H)(1 + 2\theta c \delta_H)}{2(1 - (\sin(\pi H))^2)}.$$

**Proof.** The proof of Lemma 12 is given in Appendix B. □

**Proof of Theorem 4, second part.** The expansions (4.50) and (4.56) imply (2.16), which completes the proof of Theorem 4. □

### 4.3 Proof of Theorem 5.

We shall now proceed to the proof of Theorem 5 which essentially follows the same lines as that of Theorem 4, second part. First of all, one can observe that if  $c = -1/(2\theta\delta_H)$ , then we exactly have  $a_c = a_H$ . As in the proof of Theorem 4, there exists a unique  $a_T$ , which belongs to the interior of  $D_H = ]-\infty, a_H[$  and converges to its border  $a_H$ , solution of the implicit equation

$$(4.57) \quad \Lambda'_T(a) = c = -\frac{1}{2\theta\delta_H}$$

where  $\Lambda_T$  is given by (4.36). We deduce from (4.57) that

$$(4.58) \quad T(\varphi_T + \theta\delta_H)^2 = \frac{\theta(\varphi_T + \theta p_H)}{c\varphi_T(\varphi_T - \theta)(2 + p_H)}.$$

Consequently, we infer from (4.57) and (4.58) that there exists a sequence  $(a_k)$  such that, for any  $p > 0$  and  $T$  large enough,

$$(4.59) \quad a_T = \sum_{k=0}^{2p} \frac{a_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right) \quad \text{and} \quad \varphi_T = \sum_{k=0}^{2p} \frac{\varphi_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)$$

with  $a_0 = a_H$ ,  $\varphi_0 = -\theta\delta_H$ ,  $a_1 = -(-\theta\delta_H)^{3/2}$ ,  $\varphi_1 = \sqrt{-\theta\delta_H}$ ,

$$\begin{aligned} a_2 &= -\frac{\theta\delta_H}{4}(1 + \sin(\pi H)), \\ \varphi_2 &= -\frac{1}{4}(3 + \sin(\pi H)). \end{aligned}$$

Furthermore, we have the decomposition  $\mathbb{P}(S_T \geq cT) = A_T B_T$  where  $A_T$  and  $B_T$  are respectively given by (4.42) and (4.43). Via the same lines as in the proof of the expansion (4.50), we find that for any  $p > 0$  and  $T$  large enough,

$$(4.60) \quad A_T = \exp(-TI(c))(-\theta\delta_H eT)^{1/4} \sqrt{2\delta_H \sin(\pi H)} \left[ 1 + \sum_{k=1}^{2p} \frac{\alpha_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right) \right]$$

where the coefficients  $(\alpha_k)$  may be explicitly calculated. It still remains to give the expansion of  $B_T$  which can be rewritten as

$$(4.61) \quad B_T = \mathbb{E}_T \left[ \exp(-a_T \sqrt{T} U_T) \mathbb{I}_{U_T \geq 0} \right]$$

where

$$(4.62) \quad U_T = \frac{S_T - cT}{\sqrt{T}}.$$

**Lemma 13.** *For  $c = -1/(2\theta\delta_H)$ , the distribution of  $U_T$  under  $\mathbb{P}_T$  converges, as  $T$  goes to infinity, to the distribution of  $\sigma_H N_1 + \nu_H(N_2^2 - 1)$  where  $N_1$  and  $N_2$  are two independent  $\mathcal{N}(0, 1)$  random variables and*

$$(4.63) \quad \sigma_H^2 = L''(a_H) = -\frac{1}{2(\theta\delta_H)^3},$$

$$(4.64) \quad \eta_H = \frac{1}{2(-\theta\delta_H)^{3/2}}.$$

In other words, the limit of the characteristic function of  $U_T$  under  $\mathbb{E}_T$  is

$$(4.65) \quad \Phi(u) = \frac{\exp\left(-i\eta_H u - \frac{u^2\sigma_H^2}{2}\right)}{\sqrt{1 - 2i\eta_H u}}.$$

Moreover, there exists a sequence  $(\beta_k)$  such that, for any  $p > 0$  and  $T$  large enough,

$$(4.66) \quad B_T = \sum_{k=1}^{2p} \frac{\beta_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p \sqrt{T}}\right)$$

where the sequence  $(\beta_k)$  may be explicitly calculated. For example,

$$\beta_1 = \frac{1}{4\pi a_H \eta_H} \exp\left(-\frac{1}{4}\right) \Gamma\left(\frac{1}{4}\right).$$

**Proof.** The proof of Lemma 13 is given in Appendix C.  $\square$

**Proof of Theorem 5.** The expansions (4.60) and (4.66) imply (2.19), which completes the proof of Theorem 5.  $\square$

## Appendix A: On the main asymptotic expansion.

We shall first prove the asymptotic expansion (1.9) of the normalized cumulant generating function  $\mathcal{L}_T(a, b)$ . This result was partially established by formula (5.12) of Kleptsyna and Le Breton [10]. By Girsanov's theorem,  $\mathcal{L}_T(a, b)$  can be rewritten as

$$\begin{aligned} \mathcal{L}_T(a, b) &= \frac{1}{T} \log \mathbb{E} \left[ \exp \left( a \int_0^T Q_t dY_t + bS_T \right) \right], \\ &= \frac{1}{T} \log \mathbb{E}_\varphi \left[ \exp \left( (a + \theta - \varphi) \int_0^T Q_t dY_t + \frac{1}{2}(2b - \theta^2 + \varphi^2)S_T \right) \right] \end{aligned}$$

for all  $\varphi \in \mathbb{R}$ , where  $\mathbb{E}_\varphi$  stands for the expectation after the usual change of probability

$$\frac{d\mathbb{P}_\varphi}{d\mathbb{P}} = \exp \left( (\varphi - \theta) \int_0^T Q_t dY_t - \frac{1}{2}(\varphi^2 - \theta^2)S_T \right).$$

If  $\theta^2 - 2b > 0$ , we can choose  $\varphi = \sqrt{\theta^2 - 2b}$  and  $\tau = \varphi - (a + \theta)$  which leads to

$$(A.1) \quad \mathcal{L}_T(a, b) = \frac{1}{T} \log \mathbb{E}_\varphi \left[ \exp \left( -\tau \int_0^T Q_t dY_t \right) \right].$$

By Itô formula, we also have

$$\int_0^T Q_t dY_t = \frac{1}{2} \left( l_H Y_T \int_0^T t^{2H-1} dY_t - T \right).$$

Consequently, we obtain from (A.1) that

$$(A.2) \quad \mathcal{L}_T(a, b) = \frac{\tau}{2} + \frac{1}{T} \log \mathbb{E}_\varphi \left[ \exp \left( -\frac{\tau l_H}{2} Y_T \int_0^T t^{2H-1} dY_t \right) \right].$$

Under the new probability  $\mathbb{P}_\varphi$ , the pair  $(Y_T, \int_0^T t^{2H-1} dY_t)$  is Gaussian with mean zero and covariance matrix  $\Gamma_T(\varphi)$ . Denote  $I$  and  $J$  the two matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

As soon as the matrix

$$M_T(a, b) = I + \frac{\tau l_H}{2} \Gamma_T^{1/2}(\varphi) J \Gamma_T^{1/2}(\varphi)$$

is positive definite, we deduce from (A.2) together with standard calculus on the Gaussian distribution that

$$(A.3) \quad \mathcal{L}_T(a, b) = \frac{\tau}{2} - \frac{1}{2T} \log \det(M_T(a, b)).$$

Furthermore, it was already proven by relation (5.12) of [10] that, if  $\tau > 0$

$$(A.4) \quad \det(M_T(a, b)) = \frac{1}{z_T} \left[ x_T \left( 1 + \frac{\tau}{\varphi} e^{\delta_T} \sinh(\delta_T) \right)^2 - y_T \left( 1 - \frac{\tau}{\varphi} e^{\delta_T} \cosh(\delta_T) \right)^2 \right]$$

with  $\delta_T = T\varphi/2$ ,  $x_T = I_{H-1}(\delta_T)I_{-H}(\delta_T)$ ,  $y_T = I_{1-H}(\delta_T)I_H(\delta_T)$  and

$$z_T = x_T - y_T = \frac{4 \sin(\pi H)}{\pi \varphi T}$$

where  $I_H$  is the modified Bessel function of the first kind. We refer the reader to [11] Chapter 5 for the main properties of Bessel functions. Therefore, if  $p_T = (x_T + y_T)/z_T$  and  $r_T = 2p_T e^{-T\varphi} - 1$ , we deduce from (A.4) after some straightforward calculations that

$$(A.5) \quad \begin{aligned} \det(M_T(a, b)) &= \frac{(2\varphi - \tau)^2}{4\varphi^2} + p_T \frac{\tau(2\varphi - \tau)}{2\varphi^2} e^{T\varphi} + \frac{\tau^2}{4\varphi^2} e^{2T\varphi} \\ &= \frac{\tau}{2\varphi} e^{2T\varphi} \left( 1 + \frac{(2\varphi - \tau)}{2\varphi} r_T + \frac{(2\varphi - \tau)^2}{2\varphi\tau} e^{-2T\varphi} \right). \end{aligned}$$

Consequently, we infer from (A.3) and (A.5) that

$$\begin{aligned} \mathcal{L}_T(a, b) &= -\frac{1}{2}(a + \theta + \varphi) - \frac{1}{2T} \log \left( \frac{\tau}{2\varphi} \right) - \frac{1}{2T} \log \left( 1 + \frac{(2\varphi - \tau)}{2\varphi} r_T \right) \\ &\quad - \frac{1}{2T} \log \left( 1 + \frac{(2\varphi - \tau)^2}{\tau(2\varphi + r_T(2\varphi - \tau))} e^{-2T\varphi} \right). \end{aligned}$$

In order to complete the proof of Lemma 1, it remains to show that the limiting domain  $\Delta_H$  reduces to  $\theta^2 - 2b > 0$  and  $\sqrt{\theta^2 - 2b} > \max(a + \theta; -\delta_H(a + \theta))$ . On the one hand, we already saw that our calculation is true as soon as  $\theta^2 - 2b > 0$  and  $\tau > 0$  which can be rewritten as

$$\varphi > a + \theta.$$

On the other hand, we also have the second constraint

$$(A.6) \quad 1 + \frac{(2\varphi - \tau)}{2\varphi} r_T > 0$$

leading to

$$(A.7) \quad \sqrt{\theta^2 - 2b} > -\delta_H(a + \theta).$$

As a matter of fact, it follows from the asymptotic expansion (5.11.10) of [11] for the Bessel function  $I_H$  that for all  $z \in \mathbb{C}$  with  $|z|$  large enough and  $|\arg(z)| \leq \pi/2 - \delta$  where  $\delta$  is an arbitrarily small positive number and for any  $p > 0$

$$(A.8) \quad r_H(z) = \frac{\exp(2z)}{\sin(\pi H)} \left[ 1 + \sum_{k=1}^p \frac{r_k^H}{z^k} + \mathcal{O}\left(\frac{1}{|z|^{p+1}}\right) \right].$$

Moreover, the coefficients  $(r_k^H)$  may be explicitly calculated. For example, one can check that  $r_1^H = -(2H-1)^2/4$  and  $r_2^H = (2H-1)^2(2H+1)(2H-3)/32$ . In addition, all the coefficients  $(r_k^H)$  vanish to zero if  $H = 1/2$ . Consequently,

$$(A.9) \quad r_T(a) = p_H + \frac{1}{\sin(\pi H)} \sum_{k=1}^p \frac{2^k r_k^H}{(\varphi(a))^k T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)$$

with  $p_H = (1 - \sin(\pi H))/\sin(\pi H)$ . Hence, as  $T$  tends to infinity, (A.6) reduces to  $2\varphi + (\varphi + (a + \theta))p_H > 0$  so  $\varphi(2 + p_H) > -p_H(a + \theta)$ . Finally, as  $\delta_H = p_H/(2 + p_H)$ , it clearly implies (A.7) which completes the proof of Lemma 1.  $\square$

## Appendix B: On the characteristic functions.

### B.1 Proof of Lemma 10.

If  $\Phi_T$  denotes the characteristic function of  $U_T$  under  $\mathbb{P}_T$ , it follows from (4.27) that

$$(B.1) \quad \Phi_T(u) = \exp\left(-\frac{iuc\sqrt{T}}{\sigma_c} + T\left(L_T\left(a_c + \frac{iu}{\sigma_c\sqrt{T}}\right) - L_T(a_c)\right)\right).$$

First of all, it is necessary to prove that for  $T$  large enough,  $\Phi_T$  belongs to  $L^2(\mathbb{R})$ . One can observe that, in contrast with [3], it is impossible here to make use of the Karhunen-Loëve expansion of the process  $(X_t)$ .

**Lemma 14.** *For  $T$  large enough,  $\Phi_T$  belongs to  $L^2(\mathbb{R})$ .*

**Proof.** It is a direct consequence of Proposition 19 page 29.  $\square$

We shall now establish an asymptotic expansion for the characteristic function  $\Phi_T$ , similar to that of Lemma 7.1 of [3].

**Lemma 15.** *For any  $p > 0$ , there exist integers  $q(p)$ ,  $r(p)$  and a sequence  $(\varphi_{k,l}^H)$  independent of  $p$ , such that, for  $T$  large enough*

$$(B.2) \quad \Phi_T(u) = \exp\left(-\frac{u^2}{2}\right) \left[ 1 + \frac{1}{\sqrt{T}} \sum_{k=0}^{2p} \sum_{l=k+1}^{q(p)} \frac{\varphi_{k,l}^H u^l}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{\max(1, |u|^{r(p)})}{T^{p+1}}\right) \right]$$

and the remainder  $\mathcal{O}$  is uniform as soon as  $|u| \leq sT^{1/6}$  for some positive constant  $s$ .

**Proof.** It is rather easy to see that for all  $k \in \mathbb{N}$ ,  $R_T^{(k)}(a_c) = \mathcal{O}(T^k \exp(-T/c))$ . Hence, we infer from (1.9) together with (2.8), (2.9), (2.10) that for all  $k \in \mathbb{N}$ ,

$$(B.3) \quad L_T^{(k)}(a_c) = L^{(k)}(a_c) + \frac{1}{T} H^{(k)}(a_c) + \frac{1}{T} K_T^{(k)}(a_c) + \mathcal{O}(T^k \exp(-T/c)).$$

Therefore, we find from (B.1) and (B.3) that for any  $p > 0$ ,

$$\begin{aligned} \log \Phi_T(u) &= -\frac{u^2}{2} + T \sum_{k=3}^{2p+3} \left( \frac{iu}{\sigma_c \sqrt{T}} \right)^k \frac{L^{(k)}(a_c)}{k!} \\ &\quad + \sum_{k=1}^{2p+1} \left( \frac{iu}{\sigma_c \sqrt{T}} \right)^k \frac{H^{(k)}(a_c) + K_T^{(k)}(a_c)}{k!} + \mathcal{O}\left(\frac{\max(1, u^{2p+4})}{T^{p+1}}\right). \end{aligned}$$

We deduce the asymptotic expansion (B.2) by taking the exponential on both sides, remarking that, as soon as  $|u| \leq sT^{1/6}$  some positive constant  $s$ , the quantity  $u^l/(\sqrt{T})^k$  remains bounded in (B.2).  $\square$

**Proof of Lemma 10.** It follows from Parseval's formula that  $B_T$ , given by (4.31), can be rewritten as

$$(B.4) \quad B_T = \frac{1}{2\pi a_c \sigma_c \sqrt{T}} \int_{\mathbb{R}} \left( 1 + \frac{iu}{a_c \sigma_c \sqrt{T}} \right)^{-1} \Phi_T(u) du.$$

For some positive constant  $s$ , set  $s_T = sT^{1/6}$ . We can split  $B_T = C_T + D_T$  with

$$(B.5) \quad C_T = \frac{1}{2\pi a_c \sigma_c \sqrt{T}} \int_{|u| \leq s_T} \left( 1 + \frac{iu}{a_c \sigma_c \sqrt{T}} \right)^{-1} \Phi_T(u) du,$$

$$(B.6) \quad D_T = \frac{1}{2\pi a_c \sigma_c \sqrt{T}} \int_{|u| > s_T} \left( 1 + \frac{iu}{a_c \sigma_c \sqrt{T}} \right)^{-1} \Phi_T(u) du.$$

From now on, we claim that for some positive constant  $\nu$ ,

$$(B.7) \quad |D_T| = \mathcal{O}(\exp(-\nu T^{1/3})).$$

As a matter of fact, it follows from (B.1) that

$$|\Phi_T(u)| \leq \exp \left( T \left( L_T \left( a_c + \frac{iu}{\sigma_c \sqrt{T}} \right) - L_T(a_c) \right) \right).$$

We also deduce from (2.8) that  $L(a_c) > 0$  and thus, using Proposition 20, we find that

$$|\Phi_T(u)| \leq \exp(-TL(a_c)) \exp \left( -\frac{T|u|}{8\sqrt{2}\varphi(a_c)} \left( 1 + \frac{2|u|}{\varphi^2(a_c)} \right)^{-3/4} \right)$$

which leads to (B.7). Finally, we deduce (4.32) from (B.2) and (B.5) together with standard calculus on the  $\mathcal{N}(0, 1)$  distribution.  $\square$

## B.2 Proof of Lemma 12.

If  $\Phi_T$  stands for the characteristic function of  $U_T$  under  $\mathbb{P}_T$ , we have from (4.41)

$$(B.8) \quad \Phi_T(u) = \exp \left( -iuc + T \left( L_T \left( a_T + \frac{iu}{T} \right) - L_T(a_T) \right) \right).$$

As in the proof of Lemma 10, it follows from Proposition 19 page 29 that for  $T$  large enough,  $\Phi_T$  belongs to  $L^2(\mathbb{R})$ . We shall now propose an asymptotic expansion for  $\Phi_T$ , slightly different from that of Lemma 7.2 of [3].

**Lemma 16.** *For any  $p > 0$ , there exist integers  $q(p)$ ,  $r(p)$  and a sequence  $(\varphi_{k,l,m}^H)$  independent of  $p$ , such that, for  $T$  large enough*

$$\Phi_T(u) = \Phi(u) \exp \left( -\frac{\sigma_H^2 u^2}{2T} \right) \left[ 1 + \sum_{k=1}^p \sum_{l=1}^{q(p)} \sum_{m=1}^{r(p)} \frac{\varphi_{k,l,m}^H u^l}{T^k (1 - 2i\nu_H u)^m} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{T^{p+1}}\right) \right]$$

where  $\Phi$  is given by (4.55),

$$\gamma_H = c - L'(a_H) = \frac{1 + 2\theta c \delta_H}{2\theta \delta_H}, \quad \sigma_H^2 = L''(a_H) = -\frac{1}{2\theta^3 \delta_H^3},$$

and

$$\nu_H = \frac{(1 - (\sin \pi H)^2)\gamma_H}{1 - (\sin \pi H)^2 - (2H - 1)^2 \sin \pi H (1 + 2\theta c \delta_H)}.$$

Moreover, the remainder  $\mathcal{O}$  is uniform as soon as  $|u| \leq sT^{2/3}$  for some positive constant  $s$ .

**Remark 5.** One can observe in this asymptotic expansion the limiting  $\chi^2$  distribution  $\Phi$  together with an independent centered Gaussian distribution with small variance  $\sigma^2/T$ .

**Proof.** First of all, we deduce from (4.33) that

$$(B.9) \quad L_T(a_T) = L(a_T) + \frac{1}{T} H(a_T) + \frac{1}{T} K(a_T) + \frac{1}{T} \check{R}_T(a_T).$$

On the one hand, (2.8) implies that

$$T \left( L \left( a_T + \frac{iu}{T} \right) - L(a_T) \right) = -\frac{T \varphi_T}{2} \left( \left( 1 - \frac{iub_T}{T} \right)^{1/2} - 1 \right)$$

with  $b_T = 2/\varphi_T^2$ . Consequently, for all  $p \geq 2$

$$\exp \left( T \left( L \left( a_T + \frac{iu}{T} \right) - L(a_T) \right) \right) = \exp \left( \frac{iu \varphi_T b_T}{4} - \frac{T \varphi_T}{2} \sum_{k=2}^p l_k \left( \frac{iub_T}{T} \right)^k + \mathcal{O}\left(\frac{|u|^{p+1}}{T^p}\right) \right)$$

where  $l_k = -(2k)!/((2k-1)(2^k k!)^2)$  which leads to

$$(B.10) \quad \exp \left( -iuc + T \left( L(a_T + \frac{iu}{T}) - L(a_T) \right) \right)$$

$$= \exp\left(-iu\gamma_H - \frac{\sigma_H^2 u^2}{2T}\right) \left[1 + \sum_{k=1}^p \sum_{l=1}^{q(p)} \frac{\varphi_{k,l}^H u^l}{T^k} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{T^{p+1}}\right)\right].$$

On the other hand, we also have from (2.9) that for all  $p \geq 1$

$$\begin{aligned} \exp\left(H\left(a_T + \frac{iu}{T}\right) - H(a_T)\right) &= \left(\frac{\varphi_T - \theta}{\varphi_T - \theta(1 - iub_T/T)^{-1/2}}\right)^{1/2}, \\ &= \left(1 - \left(\frac{\theta}{\varphi_T - \theta}\right) \sum_{k=1}^p h_k \left(\frac{iub_T}{T}\right)^k + \mathcal{O}\left(\frac{|u|^{p+1}}{T^{p+1}}\right)\right)^{-1/2} \end{aligned}$$

with  $h_k = (2k)!/(2^k k!)^2$ . Hence,

$$(B.11) \quad \exp\left(H\left(a_T + \frac{iu}{T}\right) - H(a_T)\right) = \left[1 + \sum_{k=1}^p \sum_{l=1}^{q(p)} \frac{\psi_{k,l}^H u^l}{T^k} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{T^{p+1}}\right)\right].$$

Furthermore, it follows from (4.34) that for all  $p \geq 1$

$$\begin{aligned} \exp\left(K\left(a_T + \frac{iu}{T}\right) - K(a_T)\right) &= \left(\frac{2\varphi_T + (\varphi_T + \theta)p_H}{2\varphi_T + \varphi_T p_H + \theta p_H(1 - iub_T/T)^{-1/2}}\right)^{1/2}, \\ &= \left(1 + \left(\frac{\theta p_H T}{c_T}\right) \sum_{k=1}^p h_k \left(\frac{iub_T}{T}\right)^k + \frac{1}{c_T} \mathcal{O}\left(\frac{|u|^{p+1}}{T^p}\right)\right)^{-1/2} \end{aligned}$$

where  $c_T = T(2\varphi_T + (\varphi_T + \theta)p_H)$ . Therefore, if

$$d_T(u) = 1 + \frac{iu\theta p_H b_T}{2c_T}$$

we find that for all  $p \geq 2$

$$\begin{aligned} (B.12) \quad &\exp\left(K\left(a_T + \frac{iu}{T}\right) - K(a_T)\right) \\ &= \frac{1}{\sqrt{d_T(u)}} \left(1 + \left(\frac{\theta p_H T}{c_T d_T(u)}\right) \sum_{k=2}^p h_k \left(\frac{iub_T}{T}\right)^k + \frac{1}{c_T d_T(u)} \mathcal{O}\left(\frac{|u|^{p+1}}{T^p}\right)\right)^{-1/2} \end{aligned}$$

One can easily check that as  $T$  goes to infinity, the limits of  $b_T$ ,  $c_T$  and  $d_T(u)$  are respectively given by  $2/(\theta\delta_H)^2$ ,  $-(2 + p_H)/(1 + 2\theta c\delta_H)$  and  $1 - 2i\gamma_H u$ , where  $\gamma_H$  is given by (4.53). Then, we infer from (B.12) that for all  $p \geq 2$

$$\begin{aligned} (B.13) \quad &\exp\left(K\left(a_T + \frac{iu}{T}\right) - K(a_T)\right) \\ &= \frac{1}{\sqrt{1 - 2i\gamma_H u}} \left[1 + \sum_{k=1}^p \sum_{l=1}^{q(p)} \sum_{m=1}^{r(p)} \frac{\Psi_{k,l,m}^H u^l}{T^k (1 - 2i\nu_H u)^m} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{T^{p+1}}\right)\right]. \end{aligned}$$

Now, in contrast with [3], the remainder term  $\check{R}_T$  plays a prominent role that can't be neglected. Let  $\xi_T = T(\varphi_T + \theta)(r(a_T) - p_H)/c_T$  and

$$\xi_T(u) = \frac{T}{c_T} \left(\varphi_T + \theta \left(1 - \frac{iub_T}{T}\right)^{-1/2}\right) \left(r_T\left(a_T + \frac{iu}{T}\right) - p_H\right).$$

One can observe that  $\xi_T$  and  $\xi_T(u)$  share the same limit

$$\lim_{T \rightarrow \infty} \xi_T(u) = \xi_H = \frac{2(1 - \delta_H)(1 + 2\theta c\delta_H)r_1^H}{\delta_H(2 + p_H)\sin(\pi H)} = -\frac{(2H - 1)^2 \sin(\pi H)(1 + 2\theta c\delta_H)}{1 - (\sin(\pi H))^2}.$$

In addition, it follows from (4.35) that

$$\begin{aligned} \exp(\check{R}_T(a_T)) &= \left( \frac{c_T}{c_T + c_T \xi_T} \right)^{1/2} \left[ 1 + \mathcal{O}(T \exp(2\theta T \delta_H)) \right], \\ &= (1 + \xi_T)^{-1/2} \left[ 1 + \mathcal{O}(T \exp(2\theta T \delta_H)) \right]. \end{aligned}$$

Moreover, we also have

$$\exp\left(\check{R}_T\left(a_T + \frac{iu}{T}\right)\right) = \left( \frac{d_T(u) + e_T(u) + \xi_T(u)}{d_T(u) + e_T(u)} \right)^{-1/2} \left[ 1 + \mathcal{O}(T \exp(2\theta T \delta_H)) \right]$$

where

$$e_T(u) = \frac{\theta p_H T}{c_T} \left( \left(1 - \frac{iub_T}{T}\right)^{-1/2} - 1 - \frac{iub_T}{2T} \right).$$

Therefore, via the same lines as in the proof of (B.13), we find that for all  $p \geq 2$

$$\begin{aligned} (B.14) \quad &\exp\left(\check{R}_T\left(a_T + \frac{iu}{T}\right) - \check{R}_T(a_T)\right) \\ &= \frac{\sqrt{1 - 2i\gamma_H u}}{\sqrt{1 - 2i\nu_H u}} \left[ 1 + \sum_{k=1}^p \sum_{l=1}^{q(p)} \sum_{m=1}^{r(p)} \frac{\Phi_{k,l,m}^H u^l}{T^k (1 - 2i\nu_H u)^m} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{T^{p+1}}\right) \right] \end{aligned}$$

with

$$\nu_H = \frac{\gamma_H}{1 + \xi_H} = \frac{(1 - (\sin(\pi H))^2)\gamma_H}{1 - (\sin(\pi H))^2 - (2H - 1)^2 \sin(\pi H)(1 + 2\theta c\delta_H)}.$$

Finally, Lemma 16 follows from the conjunction of (B.10), (B.11), (B.13), and (B.14).  $\square$

**Proof of Lemma 12.** Via Parseval's formula,  $B_T$ , given by (4.52), can be rewritten as

$$(B.15) \quad B_T = \frac{1}{2\pi T a_T} \int_{\mathbb{R}} \left(1 + \frac{iu}{Ta_T}\right)^{-1} \Phi_T(u) du.$$

Let  $s_T > 0$  such that  $\sqrt{T} = o(s_T)$  as  $T$  goes to infinity. We can split  $B_T = C_T + D_T$  where

$$(B.16) \quad C_T = \frac{1}{2\pi T a_T} \int_{|u| \leq s_T} \left(1 + \frac{iu}{Ta_T}\right)^{-1} \Phi_T(u) du,$$

$$(B.17) \quad D_T = \frac{1}{2\pi T a_T} \int_{|u| > s_T} \left(1 + \frac{iu}{Ta_T}\right)^{-1} \Phi_T(u) du.$$

On the one hand, we find from Proposition 19, and the fact that  $x \mapsto x(1 + x)^{-3/4}$  is increasing that for some positive constant  $\mu$ , that

$$|D_T| = \mathcal{O}\left(T(1 + T^{3/2}) \exp\left(-\frac{\mu s_T^2}{T} \left(1 + \frac{s_T^2}{T^2}\right)^{-3/4}\right)\right).$$

It clearly leads to

$$|D_T| = \mathcal{O}(\exp(-\mu s_T^2/T)).$$

On the other hand, the asymptotic expansion for  $C_T$ , which immediately leads to (4.56), follows from Lemma 25, completing the proof of Lemma 12.  $\square$

### B.3 Proof of Lemma 13.

The proof follows the same lines as the proof of Lemma 12. The most important difference is that the scale of Taylor expansion is in  $\sqrt{T}$  instead of  $T$ . Since  $\Phi_T$  is the characteristic of  $U_T$  defined by (4.62) under  $\mathbb{P}_T$  defined by (4.41), we have:

$$(B.18) \quad \Phi_T(u) = \exp \left( \frac{iu\sqrt{T}}{2\theta\delta_H} + T \left( L_T \left( a_T + \frac{iu}{\sqrt{T}} \right) - L_T(a_T) \right) \right).$$

As in the proof of Lemma 10, it follows from Proposition 19 page 29 that for  $T$  large enough,  $\Phi_T$  belongs to  $L^2(\mathbb{R})$ . We shall now propose an asymptotic expansion for  $\Phi_T$ , slightly different from that of Lemma 16.

**Lemma 17.** *For any  $p > 0$ , there exist integers  $q(p)$ ,  $r(p)$ ,  $s(p)$  and a sequence  $(\varphi_{k,l,m}^H)$  independent of  $p$ , such that, for  $T$  large enough*

$$\Phi_T(u) = \Phi(u) \left[ 1 + \sum_{k=1}^p \sum_{l=1}^{q(p)} \sum_{m=1}^{r(p)} \frac{\varphi_{k,l,m}^H u^l}{\sqrt{T}^k (1 - 2i\nu_H u)^m} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{\sqrt{T}^{p+1}}\right) \right]$$

where  $\Phi$  is given by (4.65). Moreover, the remainder  $\mathcal{O}$  is uniform as soon as  $|u| \leq sT^{1/6}$  for some positive constant  $s$ .

**Proof.** First of all, we deduce from (4.33) that

$$(B.19) \quad L_T(a_T) = L(a_T) + \frac{1}{T} H(a_T) + \frac{1}{T} K(a_T) + \frac{1}{T} \check{R}_T(a_T).$$

On the one hand, (2.8) implies that

$$T \left( L \left( a_T + \frac{iu}{\sqrt{T}} \right) - L(a_T) \right) = -\frac{T\varphi_T}{2} \left( \left( 1 - \frac{iub_T}{\sqrt{T}} \right)^{1/2} - 1 \right)$$

with  $b_T = 2/\varphi_T^2$ . Consequently, for all  $p \geq 2$

$$\exp \left( T \left( L \left( a_T + \frac{iu}{\sqrt{T}} \right) - L(a_T) \right) \right) = \exp \left( \frac{iu\varphi_T b_T \sqrt{T}}{4} - \frac{T\varphi_T}{2} \sum_{k=2}^p l_k \left( \frac{iub_T}{\sqrt{T}} \right)^k + \mathcal{O}\left(\frac{|u|^{p+1}}{\sqrt{T}^{p+1}}\right) \right)$$

where  $l_k = -(2k)!/((2k-1)(2^k k!)^2)$  which leads to

$$(B.20) \quad \exp \left( \frac{iu\sqrt{T}}{2\theta\delta_H} + T \left( L \left( a_T + \frac{iu}{\sqrt{T}} \right) - L(a_T) \right) \right)$$

$$= \exp\left(-iu\eta_H - \frac{u^2\sigma_H^2}{2}\right) \left[1 + \sum_{k=1}^p \sum_{l=1}^{q(p)} \frac{\varphi_{k,l}^H u^l}{\sqrt{T}^k} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{\sqrt{T}^{p+1}}\right)\right].$$

On the other hand, we also have from (2.9) that for all  $p \geq 1$

$$\begin{aligned} \exp\left(H\left(a_T + \frac{iu}{\sqrt{T}}\right) - H(a_T)\right) &= \left(\frac{\varphi_T - \theta}{\varphi_T - \theta(1 - iub_T/\sqrt{T})^{-1/2}}\right)^{1/2}, \\ &= \left(1 - \left(\frac{\theta}{\varphi_T - \theta}\right) \sum_{k=1}^p h_k \left(\frac{iub_T}{\sqrt{T}}\right)^k + \mathcal{O}\left(\frac{|u|^{p+1}}{\sqrt{T}^{p+1}}\right)\right)^{-1/2} \end{aligned}$$

with  $h_k = (2k)!/(2^k k!)^2$ . Hence,

$$(B.21) \quad \exp\left(H\left(a_T + \frac{iu}{\sqrt{T}}\right) - H(a_T)\right) = \left[1 + \sum_{k=1}^p \sum_{l=1}^{q(p)} \frac{\psi_{k,l}^H u^l}{\sqrt{T}^k} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{\sqrt{T}^{p+1}}\right)\right].$$

Furthermore, it follows from (4.34) that for all  $p \geq 1$

$$\begin{aligned} \exp\left(K\left(a_T + \frac{iu}{\sqrt{T}}\right) - K(a_T)\right) &= \left(\frac{2\varphi_T + (\varphi_T + \theta)p_H}{2\varphi_T + \varphi_T p_H + \theta p_H(1 - iub_T/\sqrt{T})^{-1/2}}\right)^{1/2}, \\ &= \left(1 + \left(\frac{\theta p_H \sqrt{T}}{c_T}\right) \sum_{k=1}^p h_k \left(\frac{iub_T}{\sqrt{T}}\right)^k + \frac{1}{c_T} \mathcal{O}\left(\frac{|u|^{p+1}}{\sqrt{T}^p}\right)\right)^{-1/2} \end{aligned}$$

where  $c_T = \sqrt{T}(2\varphi_T + (\varphi_T + \theta)p_H)$ . Therefore, if

$$d_T(u) = 1 + \frac{iu\theta p_H b_T}{2c_T},$$

we find that for all  $p \geq 2$

$$(B.22) \quad \begin{aligned} &\exp\left(K\left(a_T + \frac{iu}{\sqrt{T}}\right) - K(a_T)\right) \\ &= \frac{1}{\sqrt{d_T(u)}} \left(1 + \left(\frac{\theta p_H T}{c_T d_T(u)}\right) \sum_{k=2}^p h_k \left(\frac{iub_T}{\sqrt{T}}\right)^k + \frac{1}{c_T d_T(u)} \mathcal{O}\left(\frac{|u|^{p+1}}{\sqrt{T}^p}\right)\right)^{-1/2} \end{aligned}$$

One can easily check that as  $T$  goes to infinity, the limits of  $b_T$ ,  $c_T$  and  $d_T(u)$  are respectively given by  $2/(\theta\delta_H)^2$ ,  $(2 + p_H)\sqrt{-\theta\delta_H}$  and  $1 - 2i\eta_H u$ , where  $\eta_H$  is given by (4.64). Then, we infer from (B.22) that for all  $p \geq 2$

$$(B.23) \quad \begin{aligned} &\exp\left(K\left(a_T + \frac{iu}{\sqrt{T}}\right) - K(a_T)\right) \\ &= \frac{1}{\sqrt{1 - 2i\eta_H u}} \left[1 + \sum_{k=1}^p \sum_{l=1}^{q(p)} \sum_{m=1}^{r(p)} \frac{\Psi_{k,l,m}^H u^l}{\sqrt{T}^k (1 - 2i\nu_H u)^m} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{\sqrt{T}^{p+1}}\right)\right]. \end{aligned}$$

Now, in contrast with [3], the remainder term  $\check{R}_T$  plays a prominent role that can't be neglected. Let  $\xi_T = \sqrt{T}(\varphi_T + \theta)(r(a_T) - p_H)/c_T$  and

$$\xi_T(u) = \frac{\sqrt{T}}{c_T} \left(\varphi_T + \theta \left(1 - \frac{iub_T}{\sqrt{T}}\right)^{-1/2}\right) \left(r_T\left(a_T + \frac{iu}{\sqrt{T}}\right) - p_H\right).$$

One can observe that  $\xi_T$  and  $\xi_T(u)$  share the same limit

$$\lim_{T \rightarrow \infty} \xi_T(u) = 0.$$

In addition, it follows from (4.35) that

$$\begin{aligned} \exp(\check{R}_T(a_T)) &= \left( \frac{c_T}{c_T + c_T \xi_T} \right)^{1/2} \left[ 1 + \mathcal{O}\left(\sqrt{T} \exp(2\theta T \delta_H)\right) \right], \\ &= (1 + \xi_T)^{-1/2} \left[ 1 + \mathcal{O}\left(\sqrt{T} \exp(2\theta T \delta_H)\right) \right]. \end{aligned}$$

Moreover, we also have

$$\exp\left(\check{R}_T\left(a_T + \frac{iu}{T}\right)\right) = \left( \frac{d_T(u) + e_T(u) + \xi_T(u)}{d_T(u) + e_T(u)} \right)^{-1/2} \left[ 1 + \mathcal{O}\left(\sqrt{T} \exp(2\theta T \delta_H)\right) \right]$$

where

$$e_T(u) = \frac{\theta p_H T}{c_T} \left( \left(1 - \frac{iub_T}{\sqrt{T}}\right)^{-1/2} - 1 - \frac{iub_T}{2\sqrt{T}} \right).$$

Therefore, via the same lines as in the proof of (B.23), we find that for all  $p \geq 2$

$$\begin{aligned} (B.24) \quad & \exp\left(\check{R}_T\left(a_T + \frac{iu}{T}\right) - \check{R}_T(a_T)\right) \\ &= 1 + \sum_{k=1}^p \sum_{l=1}^{q(p)} \sum_{m=1}^{r(p)} \frac{\Phi_{k,l,m}^H u^l}{\sqrt{T}^k (1 - 2i\nu_H u)^m} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{\sqrt{T}^{p+1}}\right). \end{aligned}$$

Finally, Lemma 17 follows from the conjunction of (B.20), (B.21), (B.23), and (B.24).  $\square$

**Proof of Lemma 13.** Via Parseval's formula,  $B_T$ , given by (4.52), can be rewritten as

$$(B.25) \quad B_T = \frac{1}{2\pi T a_T} \int_{\mathbb{R}} \left(1 + \frac{iu}{Ta_T}\right)^{-1} \Phi_T(u) du.$$

For some positive constant  $s$ , set  $s_T = s T^{1/6}$ . We can split  $B_T = C_T + D_T$  where

$$(B.26) \quad C_T = \frac{1}{2\pi T a_T} \int_{|u| \leq s_T} \left(1 + \frac{iu}{Ta_T}\right)^{-1} \Phi_T(u) du,$$

$$(B.27) \quad D_T = \frac{1}{2\pi T a_T} \int_{|u| > s_T} \left(1 + \frac{iu}{Ta_T}\right)^{-1} \Phi_T(u) du.$$

On the one hand, we find from Proposition 19 that for some positive constant  $\mu$ ,

$$|D_T| = \mathcal{O}(\exp(-\mu s_T^2/T)).$$

On the other hand, the asymptotic expansion for  $C_T$ , which immediately leads to (4.56), following the same arguments as those of Bercu and Rouault [3, page 18].  $\square$

## Appendix C: Technical results.

### C.1 Statement of the results

The main interest of the decomposition (4.33) is given by the two following results. They show us that the different functions we deal with are holomorphs and that the behaviour of the remainder is negligable in our calculations.

**Proposition 18** *Denote*

$$\mathcal{D}_\Delta = \{a \in \mathbb{C}, \quad \operatorname{Re}(a) < a_H\}$$

and, for  $\varepsilon > 0$ ,

$$\mathcal{D}_1 = \left\{ a \in \mathbb{C}, \quad \operatorname{Re}(a) < a_H - \varepsilon(2 + \varepsilon) \frac{\theta^2 \delta_H^2}{2} \right\}.$$

Then, for  $T$  large enough, we have the following assertions.

- a) The functions  $\varphi, L_T, L, H, K$  and  $\check{R}_T$  have analytic extensions to  $\mathcal{D}_\Delta$ .
- b) The function  $(a, T) \mapsto \check{R}_T(a)$  is  $C^\infty$  on  $\mathcal{D}_\Delta \times [T_\Delta, +\infty[$ , for  $T_\Delta$  depending only on  $H$  and  $\theta$ .
- c) For  $\varepsilon > 0$ , and for all  $a \in \mathcal{D}_1$ ,

$$\sqrt{\frac{\delta_H}{(2 + p_H)(\delta_H + 1)}} \leq |\exp(H(a) + K(a))| \leq \frac{4}{\sqrt{2 + p_H}} \sqrt{\frac{1 + \varepsilon}{\varepsilon}}.$$

- d) There exists a constant  $C$  depending only on  $\theta$  and  $H$  such that for  $T$  large enough and for all  $a \in \mathcal{D}_1$ ,

$$\sqrt{\frac{1}{2} - \frac{C}{T^2 \varepsilon (\delta_H \theta)^4}} \leq |\exp(-\check{R}_T(a))| \leq C \sqrt{1 + \frac{1}{T} + \frac{1}{T \varepsilon}}.$$

**Proposition 19** For  $T$  large enough, for  $\varepsilon > 2C/(T^2(\delta_H \theta)^4)$ ,  $a \in \mathcal{D}_1 \cap \mathbb{R}$  and  $u \in \mathbb{R}$ ,

$$|\exp(T(L_T(a + iu) - L_T(a)))| \leq C(1 + T^{3/2}) \exp\left(-\frac{T|u|}{8\sqrt{2}\varphi(a)} \left(1 + \frac{2|u|}{\varphi^2(a)}\right)^{-3/4}\right)$$

and the map

$$u \mapsto \exp(T(L_T(a + iu) - L_T(a)))$$

belongs to  $L^2(\mathbb{R})$ .

**Proposition 20** As  $T$  goes to infinity and  $a \in \mathbb{R}$  such that  $a < a_H$ , we have

$$\exp(-\check{R}_T(a)) = \mathcal{O}\left(\max\left(1; \frac{-1}{T(\varphi(a) + \delta_H \theta)}\right)\right).$$

## C.2 Proofs of the results

We shall denote the principal determination of the logarithm defined on  $\mathbb{C} \setminus ]-\infty, 0]$  by

$$\log[z] = \log|z| + i\text{Arg}(z),$$

where

$$\begin{aligned}\text{Arg}(z) &= \arcsin \left[ \frac{\text{Im}(z)}{|z|} \right] && \text{if } \text{Re}(z) \geq 0, \\ &= \arccos \left[ \frac{\text{Re}(z)}{|z|} \right] && \text{if } \text{Re}(z) < 0, \text{ Im}(z) > 0, \\ &= -\arccos \left[ \frac{\text{Re}(z)}{|z|} \right] && \text{if } \text{Re}(z) < 0, \text{ Im}(z) < 0,\end{aligned}$$

**Proof of Proposition 18:** Since  $T \mapsto S_T$  is a positive increasing process, then for  $a < a_H$

$$\mathbb{E}[\exp(aS_T)] \leq \lim_{T \rightarrow +\infty} \exp(TL_T(a)) < \infty.$$

Lebesgue dominated theorem yields that  $a \mapsto \mathbb{E}[\exp(aS_T)]$  has an analytic extension to  $\{a \in \mathbb{C}, \text{Re}(a) < a_H\}$ . In order to prove Proposition 18, we have to obtain the same result for  $\varphi, L, H, K$  and  $\check{R}_T$ . The proof is split in steps. First, we study the function  $\varphi$ .

**Lemma 21** *The function  $\varphi$  has an analytic extension on  $\{a \in \mathbb{C}, \text{Re}(a) < a_H\}$ , still denoted by  $\varphi$  such that  $\text{Arg}(\varphi) \in ]-\frac{\pi}{4}, \frac{\pi}{4}[$ ,  $\text{Re}(\varphi) \in ]-\delta_H\theta, +\infty[$ ,  $\text{Im}(\varphi)(a)$  vanishes if and only if  $\text{Im}(a) = 0$ , and for  $\varepsilon > 0$ ,*

$$\inf_{a \in \mathcal{D}_1} \{\text{Re}(\varphi(a))\} > -\theta\delta_H(1 + \varepsilon).$$

For all  $a > a_H$  and  $z \in \mathbb{R}$ ,

$$(C.1) \quad \text{Re}(\varphi(a + iz) - \varphi(a)) \geq \frac{|z|}{2\sqrt{2}\varphi(a)} \left(1 + \frac{2|z|}{\varphi^2(a)}\right)^{-3/4}.$$

**Proof of Lemma 21** The properties of  $\varphi$  relies on the properties of the analytic function defined on  $\text{Re}(z) > 0$  by

$$\sqrt{1+z} = \sqrt{|1+z|} \exp \left[ \frac{i}{2} \text{Arg}(1+z) \right],$$

and the fact that  $\varphi(a) = -\delta_H\theta \sqrt{1 + \frac{2}{\delta_H^2\theta^2}(a_H - a)}$ .

Since for  $\text{Re}(z) > 0$ ,  $\text{Arg}(z+1)$  belongs to  $] -\frac{\pi}{2}, \frac{\pi}{2}[$ , then  $\text{Arg}\sqrt{1+z}$  belongs to  $] -\frac{\pi}{4}, \frac{\pi}{4}[$ . Its imaginary part is

$$\text{Im}(\sqrt{1+z}) = \frac{\sqrt{|1+z| - \text{Re}(1+z)}}{\sqrt{2}} \text{sign}(\text{Im}(z)).$$

Its real part is given by

$$\text{Re}(\sqrt{1+z}) = \frac{\sqrt{|1+z| + \text{Re}(1+z)}}{\sqrt{2}},$$

is an increasing function of  $\text{Im}(z)$  then fulfills

$$\text{Re}(\sqrt{1+z}) - 1 \geq \frac{\text{Re}(z)}{1 + \sqrt{\text{Re}(1+z)}},$$

and for all  $z$ , such that  $\text{Re}(z) > \varepsilon(2 + \varepsilon)$ ,  $\text{Re}(\sqrt{1+z}) - 1 \geq \varepsilon$ . Inequality (C.1) is a consequence of the fact that for  $x \geq 0$ ,

$$\sqrt{1 + \sqrt{1+x}} - \sqrt{2} \geq \frac{x}{4\sqrt{2}(1+x)^{\frac{3}{4}}}.$$

□

Hereafter, we study  $H$  and  $K$ . Observe that for  $a \in ]-\infty, 0]$ ,  $H(a) = \tilde{H}(\varphi(a))$  and  $K(a) = \tilde{K}(\varphi(a))$  where for  $z \in \mathbb{C}$ ,  $\text{Re}(z) > -\delta_H \theta$ ,

$$\begin{aligned}\tilde{H}(z) &= -\frac{1}{2} \log \left[ \frac{z - \theta}{2z} \right], \\ \tilde{K}(z) &= -\frac{1}{2} \log \left[ 1 + \frac{z + \theta}{2z} p_H \right].\end{aligned}$$

Then, the expected properties on  $H$  and  $K$  are some consequences of Lemma 21 and the same properties of  $\tilde{H}$  and  $\tilde{K}$  on  $\{z \in \mathbb{C}, \text{Re}(z) > -\delta_H \theta\}$  instead of  $\{a \in \mathbb{C}, \text{Re}(a) < a_H\}$ .

**Lemma 22** *The functions  $\tilde{H}$  and  $\tilde{K}$  admit analytical extensions on  $\{z, \text{Re}(z) > -\delta_H \theta\}$ . Moreover, for all  $\varepsilon > 0$ , we denote by*

$$\mathcal{D}_2 = \left\{ z \in \mathbb{C}, \text{Re}(z) > -\delta_H \theta(1 + \varepsilon), |\text{Arg}(z)| \leq \frac{\pi}{4} \right\},$$

and we have for  $z \in \mathcal{D}_2$

$$\sqrt{\frac{\delta_H}{(2 + p_H)(\delta_H + 1)}} \leq \left| \exp(\tilde{H}(z) + \tilde{K}(z)) \right| \leq \frac{4}{\sqrt{2 + p_H}} \sqrt{\frac{1 + \varepsilon}{\varepsilon}}.$$

**Proof of lemma 22 :** Using the expression of the functions  $\tilde{H}$  and  $\tilde{K}$ , it is easy to prove the analytical extensions of  $\tilde{H}$  and  $\tilde{K}$  on  $\{z \in \mathbb{C}, \text{Re}(z) > -\delta_H \theta\}$ . For  $\tilde{K}$  it is a consequence of the fact that for  $z \in \mathbb{C}$ ,  $\text{Re}(z) > -\delta_H \theta$ ,

$$\text{Re}\left(1 + \frac{z + \theta}{2z} p_H\right) \geq 1 + \frac{p_H}{2} + \frac{\theta p_H}{2\text{Re}(z)} > 1 + \frac{p_H}{2} + \frac{p_H}{-2\delta_H} = 0.$$

It remains to prove the inequality states in the lemma but, using the fact that,

$$\begin{aligned}\tilde{H}(z) + \tilde{K}(z) &= -\frac{1}{2} (\log [z - \theta] + \log [2z + (z + \theta)p_H] - 2 \log [2z]), \\ &= -\frac{1}{2} (\log [z - \theta] + \log [(2 + p_H)z + \theta p_H] - 2 \log [2z]), \\ &= -\frac{1}{2} \left( \log [z - \theta] + \log [2 + p_H] + \log \left[ z + \frac{\theta p_H}{2 + p_H} \right] - 2 \log [2z] \right), \\ &= -\frac{1}{2} (\log [2 + p_H] + \log [z - \theta] + \log [z + \delta_H \theta] - \log [4] - 2 \log [z]).\end{aligned}$$

thus

$$\exp\left(\tilde{H}(z) + \tilde{K}(z)\right) = \left(\frac{(2+p_H)(z-\theta)(z+\theta\delta_H)}{4z^2}\right)^{-1/2},$$

where  $\sqrt{z} = \sqrt{|z|} \exp(i\text{Arg}(z)/2)$ . Since for  $z \in \mathbb{C}$ ,  $|\text{Arg}(z)| \leq \frac{\pi}{4}$ ,  $|\text{Im}(z)| \leq \text{Re}(z)$ , if moreover we have  $\text{Re}(z) > -\delta_H\theta(1+\varepsilon)$ , then

$$1 \geq \left| \frac{z+\delta_H\theta}{z} \right| \geq \frac{\text{Re}(z)+\delta_H\theta}{\text{Re}(z)} \geq \frac{\varepsilon}{(1+\varepsilon)}$$

and

$$\frac{1+\delta_H}{\delta_H} \geq \left| \frac{z-\theta}{z} \right| \geq \frac{\text{Re}(z)-\theta}{2\text{Re}(z)} \geq 1.$$

We have used the fact that the function  $x \mapsto \frac{x+\delta_H\theta}{x}$  is increasing and  $x \mapsto \frac{x-\theta}{x}$  is decreasing. The third point of proposition 18 is then proved.  $\square$

Now, we focus on  $\check{R}_T$ . First, observe that for  $a \in \mathbb{C}$ , such that  $\text{Re}(a) < 0$ ,  $\check{R}_T(a) = \tilde{R}_T(\varphi(a))$ , where for  $z \in \mathbb{C}$ , such that  $\text{Re}(z) > -\delta_H\theta$ ,

$$\begin{aligned} \tilde{R}_T(z) &= -\frac{1}{2} \log \left[ 1 + \frac{(z+\theta)(\tilde{r}_T(z)-p_H)}{(2+p_H)(z+\delta_H\theta)} + \frac{(z+\theta)^2}{(2+p_H)(z-\theta)(z+\delta_H\theta)} e^{-2Tz} \right] \\ (C.2) \quad &= -\frac{1}{2} \log \left[ z(2+p_H)(z+\delta_H\theta) + z(z+\theta)(\tilde{r}_T(z)-p_H) + \frac{z(z+\theta)^2}{(z-\theta)} e^{-2Tz} \right] \\ &\quad + \frac{1}{2} \log [z(2+p_H)(z+\delta_H\theta)]. \\ (C.3) \quad &= -\frac{1}{2} \left( \log [\tilde{R}_{1,T}(z)] - \log [(2+p_H)z] - \log [z+\delta_H\theta] \right). \end{aligned}$$

where

$$\begin{aligned} \tilde{r}_T(z) &= r_H \left( \frac{Tz}{2} \right) e^{-Tz} - 1. \\ \tilde{R}_{1,T}(z) &= z(2+p_H)(z+\delta_H\theta) + z(z+\theta)(\tilde{r}_T(z)-p_H) + \frac{z(z+\theta)^2}{(z-\theta)} e^{-2Tz}. \end{aligned}$$

The properties of  $\check{R}_T$  are some consequences of lemma 21 and the following lemma.

**Lemma 23** Denote

$$\mathcal{D}_3 = \left\{ z, \text{Re}(z) > -\delta_H\theta, \text{Arg}(z) \in ] -\frac{\pi}{4}, \frac{\pi}{4} [ \right\}.$$

For  $T$  large enough and for all  $z \in \mathcal{D}_3$ ,  $\tilde{R}_{1,T}(z) \in \mathbb{C} \setminus [-\infty, 0]$ .

In fact, lemma 23 and decomposition (C.3) give us an analytical extension of  $\tilde{R}_T$  on  $\mathcal{D}_4$ . Moreover, there exists  $T_4$  depending only on  $H$  and  $\theta$  such that the function

$$(z, T) \mapsto \frac{\tilde{R}_{1,T}(z)}{(2+p_H)z(z+\delta_H\theta)}$$

is  $C^\infty$  and never vanishes on  $\{(z, T) \mid z \in \mathcal{D}_4, T > T_4\}$  and is  $C^\infty$  with respect to  $(z, T)$ . Since,  $\check{R}_T = \tilde{\check{R}}_T(\varphi)$ , then  $(a, T) \mapsto \check{R}_T(a)$  is  $C^\infty$  on  $T > T_4$  and  $\{a, \operatorname{Re}(a) < a_H\}$ . The second point of proposition 18 is then proved.  $\square$

**Proof of Lemma 23.** We recall that for  $z \in \mathbb{C}$  such that  $\operatorname{Arg}(z) \in ] -\frac{\pi}{4}, \frac{\pi}{4} [$

$$r_H(z) - 1 = \frac{\exp(2z)}{\sin(\pi H)} \left( 1 - \frac{(2H-1)^2}{4z} + \frac{1}{z^2} \sin(\pi H) F(z) \right)$$

where  $F$  is a continuous bounded function. Then, for  $z \in \mathcal{D}_3$ ,

$$\tilde{r}_T(z) - p_H = \frac{r_1}{Tz} + \frac{1}{T^2 z^2} F(Tz).$$

We can exhibit a polynomial term from  $\tilde{\check{R}}_{1,T}(z)$

$$\begin{aligned} \tilde{\check{R}}_{1,T}(z) &= z(2 + p_H)(z + \delta_H \theta) + z(z + \theta) \left( \frac{r_1}{Tz} + \frac{1}{T^2 z^2} F(Tz) \right) + \frac{z(z + \theta)^2}{(z - \theta)} e^{-2Tz}, \\ &= z(2 + p_H)(z + \delta_H \theta) + (z + \theta) \frac{r_1}{T} + \left( \frac{z + \theta}{T^2 z} F(Tz) + \frac{z(z + \theta)^2}{(z - \theta)} e^{-2Tz} \right), \\ (\text{C.4}) \quad &= z(2 + p_H)(z + \delta_H \theta) + (z + \theta) \frac{r_1}{T} + \frac{z + \theta}{T^2 z} \left( F(Tz) + \frac{T^2 z^2(z + \theta)}{(z - \theta)} e^{-2Tz} \right). \end{aligned}$$

Let us denote

$$\begin{aligned} (\text{C.5}) \quad \tilde{P}_T(z) &= z(2 + p_H)(z + \delta_H \theta) + (z + \theta) \frac{r_1}{T}, \\ C &= \frac{1 - \delta_H}{\delta_H} \sup_{\{z \in \mathcal{D}_3\}} \left\{ \left| F(Tz) + \frac{(z + \theta)}{(z - \theta)} T^2 z^2 e^{-2zT} \right| \right\} < +\infty. \end{aligned}$$

Observe that for  $z \in \mathcal{D}_3$ , on the one hand,

$$\begin{aligned} |Im(\tilde{P}_T(z))| &= |Im(z)| \left[ (2 + p_H)(2\operatorname{Re}(z) + \delta_H \theta) + \frac{r_1}{T} \right] \\ &\geq |Im(z)| \left[ \frac{(2 + p_H)}{2} (-\delta_H \theta) + \frac{r_1}{T} \right]. \end{aligned}$$

Then, for  $z \in \mathbb{C}$ , such that

$$\operatorname{Im}(z) > \frac{4}{2 + p_H} \frac{1}{-\delta_H \theta} \left[ \frac{C}{T^2} + \frac{-r_1}{T} \right],$$

$|Im(\tilde{R}_{1,T}(z))| > 0$ . On the other hand,

$$\begin{aligned} Re(\tilde{P}_T(z)) &= (2 + p_H)Re(z)(Re(z) + \delta_H \theta) + \frac{\theta r_1}{T} Re(z) - (2 + p_H)Im(z)^2, \\ &\geq -\delta_H \theta^2 \frac{r_1}{T} - (2 + p_H)Im(z)^2, \end{aligned}$$

Then, for  $T$  large enough, for all  $z \in \mathcal{D}_3$  such that if

$$\operatorname{Im}(z) > \frac{4}{2 + p_H} \frac{1}{-\delta_H \theta} \left[ \frac{C}{T^2} + \frac{-r_1}{T} \right],$$

then  $\operatorname{Re}(\tilde{\tilde{R}}_{1,T}(z)) > 0$ . We are allowed to conclude that for  $T$  large enough, for all  $z \in \mathcal{D}_3$ ,  $\tilde{\tilde{R}}_{1,T}(z) \in \mathbb{C} \setminus [-\infty, 0]$ .  $\square$

It remains to prove point 4 of proposition 18. This is given by the following lemma.

**Lemma 24** Denote

$$\mathcal{D}_4 = \{z \in \mathbb{C} \text{ , } \operatorname{Re}(z) > -\delta_H \theta(1 + \varepsilon)\}.$$

Then, there exist a constant  $C$  depending only on  $\theta$  and the index  $H$ , such that for  $T$  large enough, and  $z \in \mathcal{D}_4$ ,

$$\sqrt{\frac{1}{4}} \leq \left| \exp(-\tilde{\tilde{R}}_T(z)) \right| \leq \sqrt{C \left( 1 + \frac{1}{T} + \frac{1}{T\varepsilon} \right)}.$$

**Proof of lemma 24 :** Observe that

$$\left| \exp(-\tilde{\tilde{R}}_T(z)) \right| = \sqrt{\frac{|\tilde{\tilde{R}}_{1,T}(z)|}{(2 + p_H)|z||z + \delta_H \theta|}}.$$

From (C.4) and (C.5),

$$\frac{|\tilde{P}_T(z)|}{(2 + p_H)|z||z + \delta_H \theta|} - \frac{C}{T^2} \leq \frac{|\tilde{\tilde{R}}_{1,T}(z)|}{(2 + p_H)|z||z + \delta_H \theta|} \leq \frac{|\tilde{P}_T(z)|}{(2 + p_H)|z||z + \delta_H \theta|} + \frac{C}{T^2}.$$

On the one hand, using the very definition of  $\tilde{P}_T$ ,

$$\frac{\tilde{P}(z)}{(2 + p_H)z(z + \delta_H \theta)} = \frac{1}{z} \left[ z + \frac{r_1}{T} + \frac{\theta r_1(1 - \delta_H)}{T(z + \delta \theta)} \right]$$

and since  $r_1 \theta > 0$ , then for  $z \in \mathbb{C}$ ,  $\operatorname{Re}(z) > -\delta_H \theta$ , for  $T > 2r_1 \theta^{-1} \delta_H^{-1}$

$$\left| \frac{\tilde{P}(z)}{(2 + p_H)z(z + \delta_H \theta)} \right| \geq \frac{1}{|z|} \sqrt{\left[ \operatorname{Re}(z) + \frac{r_1}{T} \right]^2 + \operatorname{Im}(z)^2} \geq \frac{1}{2}.$$

On the other hand,

$$\frac{\tilde{P}(z)}{(2 + p_H)z(z + \delta_H \theta)} = 1 + \frac{r_1}{T} \frac{1}{(2 + p_H)z} + \frac{r_1 \theta(1 + \delta_H)}{T} \frac{1}{(2 + p_H)z(z + \delta_H \theta)}$$

and

$$\left| \frac{\tilde{P}(z)}{(2 + p_H)z(z + \delta_H \theta)} \right| \leq 1 + \frac{r_1}{\theta \delta_H T (2 + p_H)} + \frac{r_1 \theta(1 + \delta_H)}{T} \frac{1}{(2 + p_H) \delta_H^2 \theta^2 \varepsilon}$$

$\square$

The proof of Proposition 18 follows from the conjunction of Lemmas 21 to 24.  $\square$

**Proof of Proposition 19 :** Using the decomposition (4.33) page 13, the third point of Proposition 18, the fact that for  $a \in \mathcal{D}_1$ ,

$$0 < \inf_{u \in \mathbb{R}} \{\exp(\check{R}_T(a + iu))\} \leq \sup_{u \in \mathbb{R}} \{\exp(\check{R}_T(a + iu))\} < \infty.$$

and since  $\operatorname{Re}(a + iu) = \operatorname{Re}(a)$ , we only have to bound

$$u \mapsto \exp(T(L(a + iu) - L(a))).$$

The bound is clearly given by inequality (C.1).  $\square$

**Proof of Proposition 20 :** Recall that, from Lemma 23,  $\check{R}_T(a) = \tilde{\check{R}}_T(\varphi(a))$  with  $\tilde{\check{R}}_T$  given in (C.2). Moreover, there exists  $0 < z_T^- < z_T^+ < -\delta_H \theta$  such that

$$\tilde{\check{R}}_{1,T}(z) - (2 + p_H)(z - z_T^-)(z - z_T^+) = \mathcal{O}\left(\frac{1}{T^2}\right).$$

Since,  $0 < z_T^- < z_T^+ < -\delta_H \theta$ , then for all  $z \in ]-\delta_H, +\infty[$

$$\left| \frac{(2 + p_H)(z - z_T^-)(z - z_T^+)}{(2 + p_H)z(z - \delta_H \theta)} \right| \leq 1.$$

Consequently,

$$\begin{aligned} \exp(-\check{R}_T(a)) &\leq \sqrt{1 + \frac{\tilde{\check{R}}_{1,T}(\varphi(a)) - (2 + p_H)(\varphi(a) - z_T^-)(\varphi(a) - z_T^+)}{(2 + p_H)\varphi(a)(\varphi(a) + \delta_H \theta)}}, \\ &= \mathcal{O}\left(\max\left(1; \frac{-1}{T(\varphi(a) + \delta_H \theta)}\right)\right), \end{aligned}$$

which ends the proof of Proposition 20.  $\square$

### C.3 A contour integral for the Gamma function.

In order to obtain an asymptotic expansion for  $B_T$ , it is necessary to make use of the following lemma which slightly extend Lemma 7.3 of [3]. First of all, denote by  $f_{a,b}$  the density function of the Gamma  $\mathcal{G}(a, b)$  distribution with parameters  $a, b > 0$ , given by

$$(C.6) \quad f_{a,b}(x) = \begin{cases} \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx) & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For all integers  $k, \ell \geq 0$ , and for all positive real numbers  $\sigma^2, \gamma, \nu$ , let

$$v_k(a, b, \ell) = \frac{2\pi\sigma^{2k}i^\ell}{2^k k! \gamma^{2k+\ell+1}} f_{a,b}^{(2k+\ell)}(1).$$

**Lemma 25** For any integers  $p > 0$  and  $\ell \geq 0$ , we have

$$(C.7) \quad \int_{\mathbb{R}} \exp\left(-i\gamma u - \frac{\sigma^2 u^2}{2T}\right) \frac{u^\ell}{(1 - 2i\nu u)^a} du = \sum_{k=0}^p \frac{v_k(a, b, \ell)}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)$$

with  $b = \gamma/(2\nu)$ .

**Proof of Lemma 25.** Denote by  $N_\sigma$  the Gaussian kernel with positive variance  $\sigma^2$

$$N_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

It is well-known that the characteristic functions of  $N_\sigma$  and  $f_{a,b}$  are respectively given by

$$\widehat{N}_\sigma(x) = \exp\left(-\frac{\sigma^2 x^2}{2}\right) \quad \text{and} \quad \widehat{f}_{a,b}(x) = \left(1 - \frac{ix}{b}\right)^{-a}.$$

Then, it follows from Rudin [15] page 177 that for all integer  $\ell \geq 0$  and for all positive real number  $a, b, \tau$

$$(C.8) \quad \int_{\mathbb{R}} \exp\left(-ivx - \frac{\tau^2 v^2}{2}\right) v^\ell \widehat{f}_{a,b}(v) dv = 2\pi i^\ell f_{a,b} * N_\tau^{(\ell)}(x).$$

Via the same lines as in [3] page 17, it is not hard to see that for any  $p > 0$

$$(C.9) \quad f_{a,b} * N_\tau^{(\ell)}(x) = \sum_{k=0}^p \frac{\tau^{2k}}{2^k k!} f_{a,b}^{(2k+\ell)}(x) + \mathcal{O}(\tau^{2(p+1)}).$$

Hence, we deduce from the conjunction of (C.8) and (C.9) with  $\tau^2 = \sigma^2/(T\gamma^2)$  that

$$(C.10) \quad \int_{\mathbb{R}} \exp\left(-ivx - \frac{\sigma^2 v^2}{2T\gamma^2}\right) v^\ell \widehat{f}_{a,b}(v) dv = 2\pi i^\ell \sum_{k=0}^p \frac{\sigma^{2k}}{2^k k! \gamma^{2k} T^k} f_{a,b}^{(2k+\ell)}(x) + \mathcal{O}\left(\frac{1}{T^{p+1}}\right).$$

Finally, by taking the values  $x = 1$  and  $b = \gamma/(2\nu)$  together with the change of variables  $u = v/\gamma$  in (C.10), we find that

$$(C.11) \quad \int_{\mathbb{R}} \exp\left(-i\gamma u - \frac{\sigma^2 u^2}{2T}\right) \frac{u^\ell}{(1 - 2i\nu u)^a} du = \sum_{k=0}^p \frac{v_k(a, b, \ell)}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)$$

which completes the proof of Lemma 25.  $\square$

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